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Half-integer winding number solutions to the Ginzburg–Landau–Higgs equations

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Abstract

New solutions to the Abelian $U(1)$ Higgs model, corresponding to vortices of integer and half-integer winding number bound onto the edges of domain walls and possibly surrounded by annular current flows, are described. Independently of their stability issue, the existence of these states could have interesting consequences in different physical contexts.

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1. Introduction

The pivotal role played by what is now called the Abelian $U(1)$ Higgs model in the progress and the stimulus for new ideas in many areas of physics over the second half of the twentieth century need not be emphasized [1]. Ranging from phase transitions in condensed matter systems displaying quantum coherence phenomena such as superconductivity, through elementary particle physics with its issues of spontaneous and dynamical symmetry breaking, the origin of mass, colour confinement and the dual Meissner effect, to cosmology and the evolution of the universe with the formation of textures of different dimensionalities, the basic concepts of the Abelian Higgs model and its extensions are wide ranging. With Dirac's monopole, magnetic vortices were also the first example illustrating the importance of topology in classical and quantum field theories.

In this latter respect, it seems to be *de facto* a well accepted working assumption (albeit not established in a strict mathematical sense) that all topologically nontrivial solutions to the Ginzburg–Landau–Higgs (GLH) equations are magnetic vortices of integer winding number in the order parameter represented by a spacetime-dependent complex scalar field

$$\psi(x) = f(x)e^{i\theta(x)}. \tag{1}$$

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In a space of infinite extent, finite energy configurations must necessarily be such that $|\psi(x)|$ approaches its nonvanishing constant vacuum expectation value at infinity, while single-valuedness of $\psi(x)$ itself then allows only for a phase dependency of integer winding number L at infinity, $\mathrm{e}^{-\mathrm{i}L\phi}$ (ϕ being the angular direction in a plane locally transverse to the vortex). Continuity of $\psi(x)$ throughout space then implies, when $L \neq 0$, that $\psi(x)$ itself must vanish along at least one curve of dimension one at a finite distance determining the location in three dimensions of at least one vortex. Since such configurations carry a nonzero magnetic flux measured at infinity and directly proportional to the winding number L, they correspond to magnetic vortices. The $L = \pm 1$ solution is the celebrated Abrikosov (anti)vortex of superconductors described by the effective Ginzburg–Landau equations [2], or equivalently the Nielsen–Olesen vortex of the Abelian $U(1)$ Higgs model whose low energy effective relativistic dynamics is that of the bosonic string [3]. Higher winding number solutions with a single vortex are known as giant vortex states in superconductivity, and play an important role in the magnetization properties of mesoscopic superconductors [4–7].

The above topological argument provides the classifying scheme for all possible vortex solutions (of finite energy) to the GLH equations in two (infinite flat) dimensions, in terms of the then necessarily integer winding number of the map from the circle at planar infinity onto the $U(1)$ gauge group of phase transformations of the scalar field $\psi(x)$, the homotopy classes of such maps corresponding to the first homotopy group $\pi_1(U(1)) = \mathbb{Z}$. Under both the assumptions of finite energy and of an arbitrary collection of discrete zeroes of arbitrary positive integer degree in the order parameter $\psi(x)$, it has been shown [8] that for a specific critical value of the scalar field self-coupling $\lambda_0 > 0$, a critical value here denoted by $\lambda_0 = \lambda_c$, all solutions to these equations correspond to a collection of such magnetic vortices each of whose positive or negative integer winding number equals, in absolute value, the degree of the zero in $\psi(x)$ associated to that vortex. Furthermore, still precisely for that critical value λ_c, the energy of each of these solutions saturates the Bomogol'nyi–Prasad–Sommerfeld (BPS) lower bound [9, 10] proportional to its total integer winding number, showing that they satisfy, in fact, first-order self-dual differential equations from which their second-order equations of motion follow, a tell-tale sign for some underlying supersymmetry.

The stability of these magnetic vortices in the infinite plane against fluctuations *within the same topological classes of configurations* has also been studied for arbitrary values of the scalar self-coupling [11, 12]. For a coupling less than the critical one, $\lambda_0 < \lambda_c$, and a specified integer winding number L, all such giant vortex states have been shown to be stable, thus suggesting that an arbitrary collection of vortices of total winding number L would collapse into a single giant vortex with that winding number L. In contradistinction, for a coupling larger than the critical one, $\lambda_0 > \lambda_c$, only the $L = \pm 1$ fundamental vortices are stable against the considered fluctuations, while giant vortices with $|L| \geqslant 2$ fall apart into a collection of $|L|$ individual fundamental vortices each of winding number $\mathrm{sign}(L)$. Finally, specifically at the critical coupling, $\lambda_0 = \lambda_c$, all configurations corresponding to all possible same-sign integer partitions of the total winding number L are of equal energy, since they all saturate the same BPS lower bound irrespective of the relative positions of the vortices [9, 10, 13–15]. The physical understanding of these properties stems from a subtle interplay between, on the one hand, the repulsive (resp. attractive) magnetic force acting between vortices whose winding numbers are of the same (resp. opposite) sign (much like the force between parallel or antiparallel magnetic dipoles) and, on the other hand, the attractive scalar force related to the tendency of the system to relax to configurations whose condensate value $|\psi(x)|$ is as close as possible to the constant nonvanishing expectation value which minimizes the scalar field potential energy (the ratio of the latter form of energy density to the former magnetic energy density is precisely given by the scalar self-coupling). In particular exactly at the critical coupling, $\lambda_0 = \lambda_c$, the two types of forces balance each other, and vortices do not interact with one another whatever their relative positions [15]. These properties also explain why for type II superconductors, whose self-coupling is by definition larger than the critical value, $\lambda_0 > \lambda_c$, vortices organize themselves into a triangular lattice of Abrikosov vortices, whose lattice constant is a function of the scalar self-coupling, as is beautifully confirmed experimentally [16, 17].

In spite of the elegant physical insight offered by these results, the implicit assumptions on which they rely may not be satisfied by all solutions to the GLH equations, including those of finite energy. It is only in the case of the infinite plane that the restriction to configurations of finite energy implies that $|\psi(x)|$ must reach its vacuum expectation value at infinity in a continuous manner in all angular directions, and it is this latter fact together with single-valuedness of $\psi(x)$ which then requires the total winding number to be integer. However, as soon as planar domains of finite spatial extent are considered (a situation which is of interest not only to actual superconductors but also to other systems with phase transitions or topological states confined to finite volumes) the value of $|\psi(x)|$ on the boundary need no longer be the vacuum expectation value, thereby allowing *a priori* even vanishing values, and thus also possibly circumventing the apparent restriction to integer winding numbers only. What then becomes of the above topological classification?

Furthermore, another assumption always implicit in the above results [8] is that the order parameter $\psi(x)$ is such that both the corresponding function $f(x)$ as defined in (1) is positive everywhere, and that if it does vanish, it does so only at discrete points in the plane at which the phase value $\theta(x)$ is then not defined. However, even though any nonzero complex number may always be parametrized as in (1) in terms of its positive amplitude and its phase defined only modulo 2π, when it comes to a single-valued continuous complex *function* defined on the plane (or any Riemann surface) and parametrized as in (1), there appears a nontrivial correlation between the sign of the real function $f(x)$ (which may possibly change) and the lack of single-valuedness inherent to the phase parametrization in terms of $\theta(x)$, this correlation being best highlighted by considering the transport of the single-valued order parameter $\psi(x)$ along any closed *finite* contour. This last remark is topological by nature, and thus provides the means to address the question raised above with regards to a topological classification of solutions in finite planar domains. Note that this approach takes the contour of the circle at planar infinity used in the usual argument, to bring it back at any point in the plane at a finite distance, thereby providing a finer-grained tool to assess the topological properties of solutions to the GLH equations.

As a matter of fact, it was recently pointed out [18] that in finite planar domains, beyond the usual vortex configurations, the GLH equations also possess annular vortex solutions of finite energy and integer winding number, whose order parameter $\psi(x)$ vanishes not only at a given point, but also on a series of concentric closed curves surrounding that point as well as one another in an almost regular radial pattern and at which the function $f(x)$ does alternate in sign[2]. Each such successive annulus is related to a nonvanishing closed current flow always running in the same direction, responsible for part of the total magnetic flux and whose kinetic energy contributes to the total energy of the configuration an almost identical amount, thereby leading to an infinite value in the case of an annular vortex in the infinite plane. These specific properties of annular vortices explain why they could not be uncovered through the general theorems [8] restricted to only finite energy solutions in the infinite plane and such that the function $f(x)$ always remains positive and vanishes only at discrete points with some integer degree.

The above topological argument must also take account of the built-in $U(1)$ local gauge

[2] The dynamical and thermodynamical stability of these annular vortices is an open problem.

invariance of the GLH equations, thus compounding even further the issue of the lack of single-valuedness of the function $\theta(x)$ for nonzero winding number in correlation with the sign of the function $f(x)$. As we shall see, in the same way that, for $L \neq 0$, $\theta(x)$ defines a multicovering of the plane as indicated for instance by its asymptotic value at infinity $\theta(x) \simeq -L\phi$, $f(x)$ defines a double-sheeted covering of the plane, with the order parameter $\psi(x) = f(x)\mathrm{e}^{\mathrm{i}\theta(x)}$ remaining nonetheless continuous, regular and single-valued throughout spacetime. Thus, for instance, a single giant vortex with $L \neq 0$ is associated to a degenerate double covering of the plane in which the two sheets meet at a single point (the position of the vortex, at which f varies as $u^{|L|}$ with u measuring the distance to the vortex axis), while the double-sheeted covering remains nevertheless regular, continuous and differentiable everywhere, the function f having opposite signs on each sheet.

This paper provides a topological classification of solutions to the GLH equations, which should also prove to be complete being based on the fine-grained topological consideration mentioned above. When accounting for the possibility that the real function $f(x)$ may take both negative and positive values in a continuous fashion, beyond the ordinary and annular vortex configurations of integer winding number L there also exist vortices of half-integer winding number L such that the order parameter function $f(x)$ changes sign an odd number of times when transported around some given closed contours surrouding such vortices. More specifically, it will be shown that vortices of half-integer winding number (half-integer vortices, for short) have the particularity of being bound onto the edges of an odd number of two-dimensional domain walls (when viewed in three dimensions) which end on such vortices and inside which the order parameter $\psi(x)$ vanishes on a two-dimensional surface. The same type of configuration may also occur for integer vortices (of integer winding number), in which case the number of merged domain walls must be even. Usual isolated integer vortex configurations are a particular case of the latter type, with a vanishing number of domain walls ending on each of the vortices. Thus, when viewed in a plane locally transverse to any such solution, integer and half-integer vortices are characterized by having a vanishing order parameter not only at the position of each vortex but also along an even or odd number of continuous lines emanating from the vortex itself, in contrast with the *assumptions* of the usual general theorems. Furthermore, in the case of a finite planar domain, such vortices bound onto the edges of domain walls may also be surrounded by an annular pattern of successive closed current flows extending up to the boundary, in the same manner as for isolated integer vortices.

Vortices bound onto the edges of domain walls display another interesting property. Even though the order parameter $\psi(x)$ vanishes inside the domain wall, close to the edge onto which the vortex is bound a nonvanishing closed current flow must quantum tunnel through the domain wall, in order to entirely surround the location of the vortex and contribute the required amount of magnetic flux. In the context of superconductors, this phenomenon is reminiscent of the Josephson effect for S–I–S junctions [16, 17]. What is a distinctive feature of vortices bound onto the edges of domain walls, however, is that such a quantum tunnel current appears within a same single superconductor simply across a surface of vanishing order parameter whose location within the material could be anywhere and may also vary in time.

Since domain walls are regions of space in which the order parameter $|\psi(x)|$ takes values different from its vacuum expectation value, they necessarily contribute condensation energy. In first approximation and for sufficiently extended domain walls, this contribution is necessarily proportional to their length[3]. Indeed, their thickness is essentially constant since it is governed by a specific parameter of the GLH equations, namely the superconducting

[3] A property reminiscent of the close-to-linear quark potential in QCD, as S Vandoren has pointed out.

coherence length ξ, or the inverse of the Higgs boson mass M_h in a particle physics parlance (see section 2). Consequently, in the case of spatially bounded domains, domain walls may extend all the way to the boundary of the domain and lead to finite energy configurations nonetheless. Conversely, in the case of a system of infinite spatial extent, domain walls must necessarily be of finite extent. Hence in such a case, domain walls must always end on some integer or half-integer vortex within the infinite volume of the sample.

Such considerations imply the following general picture. The basic entities in terms of which solutions to the GLH equations may be built are, on the one hand, 1/2-domain walls, and on the other hand, annular current flows. By a 1/2-domain wall is meant a domain wall which ends on a half-integer vortex of winding number $L = 1/2$ or $L = -1/2$, namely a 1/2-vortex, but whose other edge may or may not be bound to another 1/2-vortex as the case may be. By an annular current flow is meant a closed current flow carrying no winding number, which may be placed around a given vortex but then in combination with other such annular flows in the radially outward concentric pattern described previously, and which then extends all the way to the boundary of the spatial domain (such configurations are excluded in the case of an unbounded domain since their energy would be infinite). Any possible vortex configuration may be viewed as being constructed from these basic building blocks, in exactly the same way that isolated giant vortices with $|L| \geqslant 2$ are viewed as a collection of $|L|$ individual Abrikosov–Nielsen–Olesen (ANO) vortices stacked on top of one another. This is also consistent with the point of view of the double-sheeted covering of the plane advocated in this paper in terms of the function $f(x)$. Such a collection of isolated giant vortices with $|L| \geqslant 2$ defines a double covering in which subsets of the points at which the two sheets cross have coalesced into single points. Similarly, the general solutions including half-integer vortices and domain walls are viewed as configurations in which some of these points and coalesced subsets of points have been split open into curves of finite length where the two sheets of the double covering of the plane cross one another, leading *in fine* to the basic entities considered above.

For instance, a $L = 3/2$ vortex is obtained from three 1/2-domain walls with $L = 1/2$ sharing a common edge and possibly stacked on top of one another. Each of these 1/2-domain walls may or may not end on another 1/2-vortex with $L = \pm 1/2$, but must do so for samples of infinite spatial extent. This is the sole constraint to be imposed. Since the total winding numbers of such bound 1/2-domain walls only take the values $L = 0, \pm 1$, this restriction in the case of unbounded domains is consistent with the restriction to only integer total winding numbers at spatial infinity as implied by the usual topological argument. Nevertheless, the assumptions of the usual theorems [8, 12] are evaded since the order parameter then vanishes not only at discrete points in the plane but also along segments of finite length, thereby allowing for the existence of these new types of configurations built from bound 1/2-domain walls. In contradistinction for a planar domain of finite extent, the total winding number need not be integer, but could also take half-integer values, since a 1/2-domain wall may extend all the way to the boundary with finite energy nonetheless. In this case, annular current flows surrouding vortices and extending up to the boundary are also possible.

The above characterization of all solutions to the GLH equations also raises the issue of the stability and lowest energy configuration of bound 1/2-domain walls, whose investigation is left for future work. Since magnetic vortices whose winding numbers are of the same (resp. opposite) sign repel (resp. attract) each other, while any deviation from its vacuum expectation value in the order parameter $\psi(x)$ induces an attractive scalar force, one should expect that the bound 1/2-domain wall binding two 1/2-vortices of opposite winding numbers $L = 1/2$ and $L = -1/2$ would collapse, in its lowest energy state, to a $L = 0$ configuration. On the other hand, the fate of bound 1/2-domain walls binding two 1/2-vortices with the same winding number $L = \pm 1/2$ is far less obvious[4]. Given the natural tension of vortices and domain walls, their energy must increase with their curvature or bending, so that the lowest energy configurations are expected to be straight vortices and domain walls. Furthermore, for sufficiently large separation their energy grows essentially linearly with the distance between bound 1/2-vortices, 1/2-domains walls behaving very much like rubber bands because of their condensation energy. Hence, it is only when the two 1/2-vortices have a nonvanishing overlap that the competition between these repulsive and attractive forces could be such that the equilibrium lowest energy configuration corresponds to a nonvanishing separation rather than a fully collapsed state. The possible overlap between vortices depends on their size, which is governed by another parameter of the GLH equations, namely the Meissner penetration length λ or the inverse of the gauge boson mass M_γ in a particle physics parlance (see section 2). In fact, the ratio of this latter length scale to that which determines the width of domain walls is related to the value of the scalar self-coupling. Consequently, there may exist some critical value $\lambda_{1/2}$ of the scalar self-coupling marking the boundary between, on the one hand, unstable bound 1/2-domain walls always collapsing to $L = \pm 1$ ANO vortices, and, on the other hand, stable bound 1/2-domain walls which would stabilize into configurations of finite length. Conversely, such a possibility also raises the stability issue for the $L = \pm 1$ ANO vortex against splitting into two $L = \pm 1/2$ vortices bound onto the edges of a bound 1/2-domain wall, as function of the scalar self-coupling. An analysis of this intriguing point is not pursued here, as it requires a full-fledged dedicated numerical analysis which is being considered elsewhere. Note that the stability analyses [8, 12] of the ANO vortex available so far do not account for the possibility of split 1/2-domain walls among configuration fluctuations, so that no conclusive statement may be made at this point one way or the other. At this stage, we refrain from invoking arguments attempting to assess whether such a critical coupling $\lambda_{1/2}$ may exist, and whether it would be smaller, larger or even exactly equal to the critical coupling λ_c for which ordinary vortices saturate the BPS bound in the infinite plane. An understanding of these stability issues could prove to be of great interest, for instance to the physical properties of magnetic vortices in superconductors, and deserves a dedicated study.

The analysis thus unravels quite an unexpectedly rich classification of vortex, domain wall and annular current flow configurations (the latter only in bounded domains) of integer and half-integer winding numbers, solving the GLH equations and generalizing the well established isolated vortex solutions. These states may all be built from basic entities bound and stacked together, namely the 1/2-domain wall and the annular current flow in the case of finite domains, and the bound 1/2-domain walls in infinite domains. Even though the bound 1/2-domain walls of winding numbers $L = \pm 1$ could well prove not to be stable against collapse into the $L = \pm 1$ ANO vortices (at least for some regime of scalar self-coupling), nonetheless the mere existence of these fluctuation modes of bound 1/2-domain walls with winding numbers $L = 0, \pm 1$ must have consequences with regards to the dynamical properties of systems described by the effective GLH equations when coupled to external sources of electromagnetic disturbances, the obvious example being superconductors.

In recent years, similar half-integer and fractional winding number topological configurations have been discussed within other models than the Abelian $U(1)$ Higgs model, which could possibly have applications in high-temperature superconductors and superfluids [19], in the QCD confinement problem [20], and for so-called Alice strings [21] in models beyond the Standard Model of particle physics. What distinguishes all these other instances of fractional winding number from the half-integer vortices discussed in this paper, is that these

[4] Note that such domain walls would repel or attract one another as well, according to whether the vortices which are bound onto their edges have winding numbers of identical or opposite signs. This may lead to curious star-like domain wall configurations.

other models possess some spontaneously broken *non-Abelian* internal symmetry, whether global or locally gauged, such that when taken around a closed contour the matter fields of these systems end up pointing in a different direction in the representation space of their internal symmetry. What is particular to the present half-integer vortices is that a similar mechanism is at work in spite of having only an Abelian $U(1)$ gauged symmetry, by carefully keeping track, when taken around closed contours at finite distance, of the possible change of sign in $f(x)$ as well as its lack of single-valuedness in correlation with that of the phase parameter $\theta(x)$, while the order parameter $\psi(x)$ itself remains single-valued throughout space.

The paper is organized as follows. In the next section, the general equations of the Abelian $U(1)$ Higgs model are considered, to lead to the topological classification of integer and half-integer vortices. Section 3 particularizes to two dimensions, namely the plane transverse to such solutions, to discuss properties related to the possible BPS bounds these solutions could saturate. Single half-vortex solutions in finite disks and annuli are then considered in section 4, including the results of some preliminary numerical simulations. Similar issues are briefly addressed in section 5 in the case of the bound $1/2$-domain wall. Finally, open questions and further outlook onto possible consequences and applications of these topological configurations of half-integer winding number of the Abelian $U(1)$ Higgs model are discussed in the Conclusions (section 6).

2. The topological analysis

2.1. The GLH equations

In particle physics units such that $\hbar = 1$, $c = 1$, $\epsilon_0 = 1$ and $\mu_0 = 1/(\epsilon_0 c^2) = 1$ (ϵ_0 and μ_0 being the vacuum electric and magnetic permittivities, respectively) the Abelian $U(1)$ Higgs model is defined by the following action[5]:

$$\int_{(\infty)} \mathrm{d}^4x^\mu \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mid (\nabla_\mu + \mathrm{i}qA_\mu)\psi \mid^2 - \frac{1}{4}\lambda_0 \left(\mid \psi \mid^2 - a^2 \right)^2 \right\} \tag{2}$$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and $\nabla_\mu \equiv \partial/\partial x^\mu$ ($\mu = 0, 1, 2, 3$; the notation ∂_μ is reserved for another purpose hereafter), while our choice of Minkowski metric signature is $(+ - - -)$. Here, q stands for the $U(1)$ charge or gauge coupling of the complex scalar field $\psi(x)$, and $a > 0$ for its expectation value with the units of a mass scale which is also the dimension of the field $\psi(x)$.

By construction, this system possesses a $U(1)$ local gauge invariance under

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\chi(x)}\psi(x) \qquad A'_\mu(x) = A_\mu(x) - \frac{1}{q}\nabla_\mu\chi(x) \tag{3}$$

where $\chi(x)$ is an arbitrary function regular throughout spacetime. This symmetry is spontaneously broken through the Higgs potential of dimensionless scalar field self-coupling $\lambda_0 > 0$, thereby leading to a massive gauge boson γ and a massive Higgs scalar h both of zero

[5] The specific case of a four-dimensional spacetime is considered, but the results of this paper may easily be extended to higher dimensions and thereby lead to domain wall solutions of integer and half-integer winding numbers of arbitrary space-like dimensionalities and bound to the edge of domain walls of one more space-like dimension. Such configurations are very much reminiscent of D-branes [22] in string and M-theory, thus raising further issues of interest which are beyond the scope of this work. In particular, in the same way that the low-energy effective dynamics of the ANO vortex is described by the Nambu–Goto action [3], the same dynamics for such domain walls and those considered in this paper is necessarily also provided by the Nambu–Goto action extended to world volumes of the appropriate dimension. In the same vein, one could also consider possible generalizations to some complex p-form coupled to a real q-form with a spontaneously broken local symmetry.

$U(1)$ charge such that

$$M_\gamma = |qa| \qquad M_h = \sqrt{2\lambda_0}\, a. \tag{4}$$

As is already implicit in most of the discussion of the Introduction, we shall rather work with another choice of units appropriate to superconductivity. A translation from one choice of units to the other is straightforward enough. Normalizing the scalar field ψ (representing the Cooper pair wave function) to its vacuum expectation value (namely the square-root of the BCS condensate in the bulk in the absence of any magnetic field), the corresponding action is given by [18]

$$\epsilon_0 c^2 \int \mathrm{d}t \int_{(\infty)} \mathrm{d}^3\vec{x} \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left(\frac{\Phi_0}{2\pi\lambda} \right)^2 \left[\left| (\nabla_\mu + \mathrm{i}\frac{q}{\hbar} A_\mu)\psi \right|^2 - \frac{1}{2\xi^2} \left(|\, \psi\, |^2 - 1 \right)^2 \right] \right\} \tag{5}$$

where now all quantities are expressed in SI units, with of course $x^\mu = (x^0, \vec{x})$ and $x^0 = ct$, $A^\mu = (\Phi/c, \vec{A})$ (Φ being the electromagnetic scalar potential) and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. In the above expression, Φ_0 stands for the unit of quantum of flux

$$\Phi_0 = \frac{2\pi\hbar}{|q|} > 0 \tag{6}$$

$q = -2e < 0$ being the Cooper pair electric charge, while λ and ξ are the temperature-dependent magnetic (and electric [23]) penetration and coherence lengths, respectively. Note how these two length scales measure the relative contributions to the effective action above, and thus also to the energy, of the magnetic and condensate energy densities. The scale ξ weighs the contribution of the gauge-covariantized gradient of any deviation in ψ from its vacuum expectation value against its potential energy, while this total condensation energy in turn is measured against the electromagnetic energy contribution through the scale λ. Note also that one has

$$\begin{aligned} \frac{E^i}{c} &= -\frac{\partial}{\partial x^i}\frac{\Phi}{c} - \frac{1}{c}\frac{\partial}{\partial t} A^i = F^{i0} \\ B^i &= \epsilon^{ijk} \left[\frac{\partial}{\partial x^j} A^k - \frac{\partial}{\partial x^k} A^j \right] = -\epsilon^{ijk} F^{jk} \qquad i, j, k = 1, 2, 3 \end{aligned} \tag{7}$$

where $\vec{E}$ and $\vec{B}$ stand for the electric and magnetic fields, respectively, and ϵ^{ijk} is the totally antisymmetric Levi–Civita tensor in three dimensions with $\epsilon^{123} = +1$.

Comparing the above two actions, the translation between the particle physics units and the 'superconductor' ones is as follows:

$$a \longleftrightarrow \frac{\Phi_0}{2\pi\lambda}, \quad \lambda_0 a^2 \longleftrightarrow \frac{1}{\xi^2}, \quad M_\gamma \longleftrightarrow \frac{1}{\lambda}, \quad M_h \longleftrightarrow \frac{\sqrt{2}}{\xi}, \quad \frac{M_h}{M_\gamma} \longleftrightarrow \sqrt{2}\kappa \tag{8}$$

where $\kappa = \lambda/\xi$ defines the Ginzburg–Landau (GL) parameter of the superconductor. In particular, the critical value λ_c for the scalar self-coupling mentioned in the Introduction corresponds to the value $\kappa = \kappa_c$ with $\kappa_c = 1/\sqrt{2}$, which, in the particle physics context, implies $M_h = M_\gamma$ corresponding to $\lambda_0 = \lambda_c = 1$.

In terms of the 'superconductor units', the GLH equations read as follows. For the GLH equation proper, we have the covariantized Ginzburg–Landau equation

$$\left[\nabla_\mu + \mathrm{i}\frac{q}{\hbar} A_\mu \right] \left[\nabla^\mu + \mathrm{i}\frac{q}{\hbar} A^\mu \right] \psi + \frac{1}{\xi^2} \left(|\psi|^2 - 1 \right) \psi = 0 \tag{9}$$

which is coupled to the inhomogeneous Maxwell equations[6]

$$\nabla^\nu F_{\nu\mu} = \mu_0 J_{\mathrm{em}\,\mu} \tag{10}$$

where the conserved electromagnetic current density $J^\mu_{\mathrm{em}} = \left(c\rho_{\mathrm{em}}, \vec{J}_{\mathrm{em}}\right)$, with $\nabla_\mu J^\mu_{\mathrm{em}} = 0$, is given by ($\psi^*$ being the complex conjugate of ψ)

$$\mu_0 J_{\mathrm{em}\,\mu} = \frac{\mathrm{i}\hbar}{2q\lambda^2}\left[\psi^*\left(\nabla_\mu\psi + \mathrm{i}\frac{q}{\hbar}A_\mu\psi\right) - \left(\nabla_\mu\psi^* - \mathrm{i}\frac{q}{\hbar}A_\mu\psi^*\right)\psi\right]. \tag{11}$$

As a matter of fact, still another choice of normalized units is far more convenient, using the canonical length scale provided by λ and the canonical magnetic field value provided by $\Phi_0/(2\pi\lambda^2)$. Let us introduce the following normalized spacetime coordinates [18, 23]

$$\tau = \frac{x^0}{\lambda} = \frac{ct}{\lambda} \qquad \vec{u} = \frac{\vec{x}}{\lambda} \tag{12}$$

such that

$$\nabla_\mu = \frac{1}{\lambda}\partial_\mu \qquad \partial_\mu = (\partial_0, \vec{\partial}) \qquad \partial_0 = \partial_\tau \equiv \frac{\partial}{\partial\tau} \qquad \vec{\partial} \equiv \frac{\partial}{\partial\vec{u}} \tag{13}$$

while in the electromagnetic sector, the normalized quantities $\vec{e}$, $\vec{b}$, $f_{\mu\nu}$, $a^\mu = (\varphi, \vec{a})$ and $J^\mu = (J^0, \vec{J}\,)$ are defined as follows:

$$\begin{aligned} &\vec{B} = \frac{\Phi_0}{2\pi\lambda^2}\,\vec{b} \qquad \frac{\vec{E}}{c} = \frac{\Phi_0}{2\pi\lambda^2}\,\vec{e} \qquad F_{\mu\nu} = \frac{\Phi_0}{2\pi\lambda^2}\,f_{\mu\nu} \\ &A^0 = \frac{\Phi_0}{2\pi\lambda}\,\varphi \qquad \vec{A} = \frac{\Phi_0}{2\pi\lambda}\,\vec{a} \end{aligned} \tag{14}$$

$$\mu_0 c\rho_{\mathrm{em}} = \frac{1}{\lambda}\frac{\Phi_0}{2\pi\lambda^2}\,J^0 \qquad \mu_0\vec{J}_{\mathrm{em}} = \frac{1}{\lambda}\frac{\Phi_0}{2\pi\lambda^2}\,\vec{J}. \tag{15}$$

In terms of these normalized variables, the GLH equation then reads

$$\begin{aligned} &\left(\partial_\mu - \mathrm{i}a_\mu\right)\left(\partial^\mu - \mathrm{i}a^\mu\right)\psi = \kappa^2\left(1 - \mid\psi\mid^2\right)\psi \\ &\text{or} \\ &\left[(\partial_\tau - \mathrm{i}\varphi)^2 - (\vec{\partial} + \mathrm{i}\vec{a})^2\right]\psi = \kappa^2\left(1 - |\psi|^2\right)\,\psi \end{aligned} \tag{16}$$

while the inhomogeneous Maxwell equations are

$$\begin{aligned} &\partial^\nu f_{\nu\mu} = J_\mu \\ &\text{or} \\ &\vec{\partial}\cdot\vec{e} = J^0 \qquad \vec{\partial}\times\vec{b} - \partial_\tau\vec{e} = \vec{J} \end{aligned} \tag{17}$$

with the Lorentz covariant electromagnetic current density $J^\mu = (J^0, \vec{J}\,)$ given by

$$J_\mu = -\frac{1}{2}\mathrm{i}\left[\psi^*(\partial_\mu\psi - \mathrm{i}a_\mu\psi) - (\partial_\mu\psi^* + \mathrm{i}a_\mu\psi^*)\psi\right] \tag{18}$$

or

$$\begin{aligned} &J^0 = -\frac{1}{2}\mathrm{i}\left[\psi^*(\partial_\tau\psi - \mathrm{i}\varphi\psi) - (\partial_\tau\psi^* + \mathrm{i}\varphi\psi^*)\psi\right] \\ &\vec{J} = \frac{1}{2}i\left[\psi^*(\vec{\partial}\psi + \mathrm{i}\vec{a}\psi) - (\vec{\partial}\psi^* - \mathrm{i}\vec{a}\psi^*)\psi\right] \end{aligned} \tag{19}$$

and satisfying the conservation equation

$$\partial_\mu J^\mu = 0 \qquad \text{or} \qquad \partial_\tau J^0 + \vec{\partial}\cdot\vec{J} = 0 \tag{20}$$

[6] The homogeneous Maxwell equations are, of course, a consequence of the definition of the field strength $F_{\mu\nu}$ in terms of the gauge potential A_μ, being the corresponding Bianchi identities.

which as always follows from the inhomogeneous Maxwell equations. Moreover, we also have

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \qquad \text{or} \qquad \vec{e} = -\vec{\partial}\varphi - \partial_\tau \vec{a} \qquad \vec{b} = \vec{\partial} \times \vec{a}. \tag{21}$$

These relations imply the homogeneous Maxwell equations as the corresponding Bianchi identities

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma} = 0 \qquad \text{or} \qquad \vec{\partial} \times \vec{e} + \partial_\tau \vec{b} = \vec{0} \qquad \vec{\partial} \cdot \vec{b} = 0 \tag{22}$$

$\epsilon^{\mu\nu\rho\sigma}$ being the four-dimensional Levi–Civita totally antisymmetric tensor with $\epsilon^{0123} = +1$.

Let us now introduce the Lorentz covariant current $j^\mu = (j^0, \vec{j})$ whose components are defined by dividing the current density $(-J^\mu)$ opposite to the electromagnetic one by the square $|\psi|^2$ of the spacetime local condensate value $|\psi|$

$$j_\mu = -\frac{J_\mu}{|\psi|^2} = \frac{1}{2}\mathrm{i}\partial_\mu \ln\left(\frac{\psi}{\psi^*}\right) + a_\mu \tag{23}$$

or

$$j^0 = -\frac{J^0}{|\psi|^2} = \frac{1}{2}\mathrm{i}\partial_\tau \ln\left(\frac{\psi}{\psi^*}\right) + \varphi \qquad \vec{j} = -\frac{\vec{J}}{|\psi|^2} = -\frac{1}{2}\mathrm{i}\vec{\partial} \ln\left(\frac{\psi}{\psi^*}\right) + \vec{a}. \tag{24}$$

In particular, one then finds [23]

$$f_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu \qquad \text{or} \qquad \vec{e} = -\partial_\tau \vec{j} - \vec{\partial} j^0 \qquad \vec{b} = \vec{\partial} \times \vec{j}. \tag{25}$$

Note that the last of these relations, $\vec{b} = \vec{\partial} \times \vec{j}$, is nothing but the celebrated second London equation which explains the Meissner effect in superconductors. On the other hand, the second of these relations, $\vec{e} = -\partial_\tau \vec{j} - \vec{\partial} j^0$, is the appropriate Lorentz covariant extension of the celebrated first London equation which reads, in our notations, $\vec{e} = -\partial_\tau \vec{j}$. Indeed, the Abelian $U(1)$ Higgs model provides an effective theory of superconductivity which is also Lorentz covariant, and in which electric and magnetic fields play roles dual to one another under Lorentz boosts. Contrary to the usual GL and London equations which are not Lorentz covariant, and do not allow for electric fields in superconductors, the covariant formalism must necessarily imply new phenomena in specific regimes where electric fields can no longer be ignored. Clearly a covariant formalism implies specific differences in the time-dependent dynamics of superconductors, but it does so already for static configurations of applied electric and magnetic fields [23].

Finally, the free energy of the system in our choice of normalized units, when subjected to some external electric and magnetic fields $\vec{e}_{\rm ext}$ and $\vec{b}_{\rm ext}$, reads

$$E = \frac{\lambda^3}{2\mu_0}\left(\frac{\Phi_0}{2\pi\lambda^2}\right)^2 \int_{(\infty)} \mathrm{d}^3\vec{u} \left\{ \left[\vec{e} - \vec{e}_{\rm ext}\right]^2 + \left[\vec{b} - \vec{b}_{\rm ext}\right]^2 \right.$$
$$\left. + \mid (\partial_\tau - \mathrm{i}\varphi)\psi \mid^2 + \mid (\vec{\partial} + \mathrm{i}\vec{a})\psi \mid^2 + \frac{1}{2}\kappa^2 \left(1 - \mid \psi \mid^2\right)^2 - \frac{1}{2}\kappa^2 \right\}. \tag{26}$$

Note that by having subtracted in the integrand the constant term $\kappa^2/2$, we have chosen to fix the zero value of the free energy at the normal–superconducting phase transition for which $\psi = 0$ and $\vec{e} = \vec{e}_{\rm ext}$, $\vec{b} = \vec{b}_{\rm ext}$. Depending on the consideration to be emphasized, especially when discussing solutions in the infinite plane, sometimes that constant term will not be included in the total energy evaluation. The physical interpretation of the overall constant factor multiplying this expression for the free energy should be clear, since it corresponds to the magnetic energy of a constant magnetic field of value $\Phi_0/(2\pi\lambda^2)$ over a volume λ^3, precisely the two scales with respect to which all quantities have been normalized.

It is useful to revert to particle physics units. From the translation rules given previously, it follows that distance scales are measured in units of $1/M_\gamma$, magnetic and electric fields in units of $M_\gamma^2/|q|$, and the energy in units of

$$\frac{\lambda^3}{2\mu_0}\left(\frac{\Phi_0}{2\pi\lambda^2}\right)^2 \longleftrightarrow \frac{M_\gamma}{2q^2} \tag{27}$$

thus displaying the well known fact that solitonic solutions to field theory equations have mass values which are proportional to the gauge boson mass and inversely proportional to the squared gauge coupling constant.

2.2. *The polar parametrization*

Let us now consider the polar decomposition of the scalar field,

$$\psi(x) = f(x)\,\mathrm{e}^{\mathrm{i}\theta(x)} \tag{28}$$

where $f(x)$ is thus a real function and $\theta(x)$ is an angular variable defined modulo π rather than 2π. Since the sign of the real function $f(x)$ cannot be constrained to be always positive, which would be an unjustifiable restriction in the case of a continuous complex function as opposed to a complex number, the arbitrariness in the overall sign of $f(x)$ translates into a definition of $\theta(x)$ only modulo π (the reason for this arbitrariness will be fully understood in section 2.4, on basis of the double-sheeted covering of the plane mentioned in the Introduction). Moreover, the lack of single-valuedness in this quantity, which is correlated to the choice of an overall sign for the real function $f(x)$, is also compounded with the $U(1)$ local gauge freedom inherent to the system, an issue which shall be addressed more closely at the end of this section and in the next one.

The conserved electromagnetic current density J^μ is then given by

$$J_\mu = -f^2 j_\mu \qquad \text{or} \qquad J^0 = -f^2\, j^0 \qquad \vec{J} = -f^2\,\vec{j} \tag{29}$$

with the conservation equation

$$\partial_\mu\left(f^2 j^\mu\right) = 0 \qquad \text{or} \qquad \partial_\tau\left(f^2 j^0\right) + \vec{\partial}\cdot\left(f^2\vec{j}\right) = 0. \tag{30}$$

In particular, the phase variable θ is determined from the equations (23) and (24), namely,

$$\partial_\mu\theta = -j_\mu + a_\mu \qquad \text{or} \qquad \partial_\tau\theta = -j^0 + \varphi \qquad \vec{\partial}\theta = \vec{j} - \vec{a}. \tag{31}$$

Separating the real and imaginary parts of the GLH equation (16), one finds that for $f \neq 0$ the imaginary part is equivalent to the current conservation equation (30) as befits a $U(1)$ gauge-invariant theory, while the real part reads

$$\left[\vec{\partial}^2 - \partial_\tau^2\right] f = \left(\vec{j}^2 - j^{0\,2}\right) f - \kappa^2(1 - f^2) f. \tag{32}$$

In order to identify an organizational principle within this set of coupled differential equations, let us substitute the covariant London equations (25) into the inhomogeneous Maxwell equations, which leads to

$$\left[\vec{\partial}^2 - \partial_\tau^2\right] j^0 = f^2 j^0 - \partial_\tau\left(\partial_\tau j^0 + \vec{\partial}\cdot\vec{j}\right) \qquad \left[\vec{\partial}^2 - \partial_\tau^2\right]\vec{j} = f^2\vec{j} + \vec{\partial}\left(\partial_\tau j^0 + \vec{\partial}\cdot\vec{j}\right). \tag{33}$$

The expected appearance of the Klein-Gordon operator $\partial_\mu\partial^\mu = [\partial_\tau^2 - \vec{\partial}^2]$ in these equations suggests to also establish a similar equation for θ, given the relations (31)

$$\left[\vec{\partial}^2 - \partial_\tau^2\right]\theta = \left(\partial_\tau j^0 + \vec{\partial}\cdot\vec{j}\right) - \left(\partial_\tau\varphi + \vec{\partial}\cdot\vec{a}\right). \tag{34}$$

In conclusion, we have obtained the following set of equations, expressed either in a manifestly covariant form or in terms of their separate spacetime components. The Lorentz covariant time-dependent GLH equation is

$$\partial_\mu \partial^\mu f = j_\mu j^\mu f + \kappa^2 (1 - f^2) f \tag{35}$$

or equivalently the equation given in (32). This GLH equation is coupled to the inhomogeneous Maxwell equations[7]

$$\partial_\nu \partial^\nu j_\mu = -f^2 j_\mu + \partial_\mu (\partial_\nu j^\nu) \tag{36}$$

or in component form the equations in (33), from which the current conservation equation (30) follows. Once this coupled set of equations for f and $j^\mu = (j^0, \vec{j})$ is solved (for a given choice of boundary conditions to be discussed presently), the electric and magnetic fields within the superconductor are determined from the Lorentz covariant London equations (25)

$$f_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu \qquad \text{or} \qquad \vec{e} = -\partial_\tau \vec{j} - \vec{\partial} j^0 \qquad \vec{b} = \vec{\partial} \times \vec{j} \tag{37}$$

from which the homogeneous Maxwell equations (22) follow. One then needs to identify gauge potentials $a^\mu = (\varphi, \vec{a})$ associated to these two fields, such that

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \qquad \text{or} \qquad \vec{e} = -\vec{\partial}\varphi - \partial_\tau \vec{a} \qquad \vec{b} = \vec{\partial} \times \vec{a} \tag{38}$$

to also determine the phase variable θ through the equations given in (31), or the second-order equation in (34), namely

$$\partial_\mu \theta = -j_\mu + a_\mu \qquad \partial_\mu \partial^\mu \theta = -\partial_\mu j^\mu + \partial_\mu a^\mu . \tag{39}$$

Finally, the free energy of the system may be expressed solely in terms of the variables f, j^0 and $\vec{j}$, the electric $\vec{e}$ and magnetic $\vec{b}$ fields within the superconductor being obtained from the covariant London equations (37),

$$E = \frac{\lambda^3}{2\mu_0} \left(\frac{\Phi_0}{2\pi\lambda^2} \right)^2 \int_{(\infty)} \mathrm{d}^3\vec{u} \left\{ \left[\vec{e} - \vec{e}_{\mathrm{ext}} \right]^2 + \left[\vec{b} - \vec{b}_{\mathrm{ext}} \right]^2 \right.$$
$$\left. + (\partial_\tau f)^2 + \left(\vec{\partial} f \right)^2 + \left(j^{0\,2} + \vec{j}^2 \right) f^2 + \frac{1}{2}\kappa^2 \left(1 - f^2 \right)^2 - \frac{1}{2}\kappa^2 \right\} . \tag{40}$$

The above relations determine the system of coupled equations to be solved, once the relevant boundary conditions are specified. Outside the superconductor, the scalar field ψ vanishes of course, and since the covariant London equations no longer apply, the complete set of inhomogeneous and homogeneous Maxwell equations then has to be considered on its own, including the possibility of applied fields or sources for these fields providing further boundary conditions. At the boundary with the superconductor, matching conditions for the electromagnetic field strength $f_{\mu\nu}$ and potential a_μ must be enforced. Furthermore, when considering variations of the action of the system with respect to the order parameter ψ, one concludes that the current $(\vec{\partial} + \mathrm{i}\vec{a})\psi$, namely the gauge-covariantized space gradient of ψ, must have a vanishing normal component at those sections of the superconductor boundary which are in contact with an isolating material. Separating the real and imaginary parts of that quantity, this condition requires that both the electromagnetic current $\vec{J} = -f^2 \vec{j}$ as well as the gradient $\vec{\partial} f$ must have a vanishing component normal to such boundaries. These different conditions provide the complete set of required boundary conditions, with one exception, however, to be discussed in the next section. Indeed, the integrated London equations, through the associated

[7] These equations may also be expressed in first-order form as $\partial^\nu f_{\nu\mu} = -f^2 j_\mu$, when using the relations between the electric and magnetic fields and the current $j^\mu = (j^0, \vec{j})$ which are defined by the covariant London equations (25).

electromagnetic flux values, provide further constraints of a global character which encode the topological vortex structure within the superconductor, which the local form of the covariant London equations in (37) cannot account for. These 'global boundary conditions' are the last essential conditions in order to define unique and gauge-invariant solutions to the GLH and Maxwell equations describing specific vortex configurations.

Until now, we have been careful to maintain manifest the Lorentz covariance properties of the system throughout. In the limit of time-independent configurations, namely static or stationary ones, the above equations reduce to the familiar Ginzburg–Landau equations for superconductors solely submitted to magnetic fields (thus with $\vec{e} = \vec{0}$, $\varphi = 0$ and $j^0 = 0$). However, also for reasons advocated in [23], when considering time-dependent configurations or time-independent situations involving electric fields as well, the usual Ginzburg–Landau equations can no longer be applicable, since in particular the usual noncovariant first London equation, $\partial_\tau \vec{j} = -\vec{e}$ (rather than the covariant one above, $\partial_\tau \vec{j} + \vec{\partial} j^0 = -\vec{e}$) implies that superconductors cannot sustain electric fields in static configurations, in contradiction with Lorentz covariance. Another way to realize that the usual GL equations cannot be Lorentz covariant is to consider the nonrelativistic limit in which the parameter c is taken to infinity. Since in SI units electric fields are measured in units of $\vec{E}/c$ relative to magnetic fields $\vec{B}$, in that limit both any time as well as any electric field dependencies decouple from the above covariant equations, which then reduce to the usual time-independent noncovariant GL equations. It is thus of great interest to explore the physical consequences in relativistic and electric regimes of the Abelian $U(1)$ Higgs model as the canonical Lorentz covariant extension of the usual Ginzburg–Landau effective field theory description of superconductivity [1]. Note, however, that the solutions which are described in this paper are static ones, in the absence of any electric fields, and thus solve, in fact, also the usual noncovariant GL equations coupled to Maxwell's equations for magnetic fields, even though these states, in fact, define Lorentz covariant configurations solving the GLH equations when subjected to arbitrary Lorentz boosts acting from their rest frame.

The advantage of having separated the system of coupled equations through the organization just described is not only that first one needs to solve only for the quantities f, j^0 and $\vec{j}$, and only then for the phase variable θ through the knowledge of the electric and magnetic fields and their associated gauge potentials[8]. More importantly perhaps, for obvious physical reasons the quantities f^2, $f^2 j^0$ and $f^2 \vec{j}$ are, in fact, gauge-invariant variables, directly amenable to physical observation, at least in principle, which is a property also shared by the electric and magnetic fields $\vec{e}$ and $\vec{b}$ only. Indeed, $U(1)$ gauge transformations of the field variables are given by

$$\psi' = \mathrm{e}^{\mathrm{i}\chi}\psi \qquad A'_\mu = A_\mu - \frac{\hbar}{q}\nabla_\mu \chi \qquad a'_\mu = a_\mu + \partial_\mu \chi \tag{41}$$

so that

$$\theta' = \theta + \chi \qquad \varphi' = \varphi + \partial_\tau \chi \qquad \vec{a}' = \vec{a} - \vec{\partial}\chi \tag{42}$$

where $\chi(\tau, \vec{u})$ is an arbitrary local function regular throughout spacetime. The real function f is defined only up to an overall sign whose choice leaves all the above equations invariant as it should, and which does not lead to any physical consequence. Nevertheless, this choice of overall sign in f is correlated to a constant shift by $\pm\pi$ in θ, again without physical consequence. This correlated arbitrariness in both f and θ is then compounded further in the case of θ because of the $U(1)$ gauge symmetries of the system, which induce arbitrary

[8] As discussed in the next section, one should keep in mind, however, the subtle connection between these two sets of variables which involves the vortex topological structure through the integrated London equations.

spacetime-dependent shifts in that variable in correlation with specific transformations in the gauge potential a_μ. In particular, note that the equations (34) and (39) display this correlation explicitly through their manifest gauge invariance.

Consequently, except for the choice of overall sign in the function f, all the gauge dependency of the system resides in the variables θ, φ and $\vec{a}$. The remaining variables, namely f (up to its overall sign), j^0, $\vec{j}$, $\vec{b}$ and $\vec{e}$ are all gauge-invariant quantities, whose determination through the above equations is decoupled from that of the gauge-dependent variables. Hence the advantage in using the procedure outlined above for the resolution of the coupled system of equations. In fact, as established in the next section, the $U(1)$ local gauge freedom is such that there always exists a choice of gauge for which the function θ is uniquely determined in terms of a specific nonregular function θ_0 which encodes the entire topological vortex structure of a given configuration, and in terms of which the gauge potential a_μ is uniquely constructed from (39) as $a_\mu = \partial_\mu\theta_0 + j_\mu$. Thus, through an appropriate choice of gauge associated to a specific vortex configuration in spacetime, only the GLH and inhomogeneous Maxwell equations (32) and (33) remain to be solved for the quantities f and $j^\mu = (j^0, \vec{j})$ subjected to the appropriate boundary conditions.

One last important point needs to be made. Even though the currents j^0 and $\vec{j}$ are gauge invariant, they are not directly amenable to physical observation, since they are so only through the electromagnetic current density $(-f^2 j^0, -f^2\vec{j})$. Hence, it is only at those points in spacetime where f does not vanish that the current $(j^0, \vec{j})$ is well-defined, while at those locations where f does vanish, it may be that this current possesses singularities just mild enough to be screened by the vanishing order parameter ψ. Indeed, this is precisely what happens not only at the location of vortices but also on the domain walls ending on them. In fact, the expressions in (23) and (24) are already indicative of the logarithmic cut structures that may appear in j^0, $\vec{j}$, f and θ at the zeroes of the order parameter ψ in a way which depends on its winding number at these points. Note also that the same remark applies to the singular character of the gauge-dependent potential a_μ at the location of vortices, a well known property typical of topological configurations for gauge fields which is made explicit in the present instance through the appearance of the singular function θ_0 in the determination of a_μ, while electric and magnetic fields are nonetheless well defined throughout spacetime, including at the location of vortices.

2.3. Configurations of integer and half-integer winding number

In order to apply the fine-grained topological analysis described in the Introduction, let us consider an arbitrary closed contour C in spacetime and measure the total electromagnetic flux through any two-dimensional surface S with C as a boundary. When normalized to the quantum of flux Φ_0, this electromagnetic flux is given by the following expression for the associated surface integral (the notation should not be confused with that for the electromagnetic scalar potential in SI units):

$$\Phi[C] = -\frac{1}{2\pi}\int_S \mathrm{d}^2u^{\mu\nu}\, f_{\mu\nu}. \tag{43}$$

If the contour C and the surface S are purely space-like, this quantity $\Phi[C]$ measures, in units of Φ_0, the total magnetic flux through that surface. In the case of a purely time-like rectangular contour with fixed boundaries in space, the quantity $\Phi[C]$ measures the electrostatic potential difference between those two points in space. The purpose of this general analysis is to make manifest once again Lorentz covariance, the discussion being eventually restricted to time-independent configurations in a planar geometry only.

By its very definition, the flux $\Phi[C]$ is not only a Lorentz scalar but also a $U(1)$ gauge-invariant quantity. Through Stokes' theorem, we have

$$\Phi[C] = -\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, a_\mu \tag{44}$$

which is recognized to be the $U(1)$ gauge-invariant Wilson loop associated to the contour C. Using now the general first-order relation in (39), we obtain

$$\Phi[C] = -\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu\theta \;-\; \frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, j_\mu = -\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu\theta \;+\; \frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \frac{1}{f^2} J_\mu. \tag{45}$$

Note that this relation, when considered for all possible spacetime contours C, provides the integrated and thus global expression of the covariant London equations (37) which in their local differential form cannot encode the topological information which we are about to characterize. Working in terms of this global form of the covariant London equations is crucial to the resolution of the GLH equations for the quantities f, j^0 and $\vec{j}$ in a way which also accounts for the topology properties of vortex configurations encoded in the functions θ, φ and $\vec{a}$. The values $\Phi[C]$ for all possible contours C in spacetime also provide the 'global boundary conditions' which connect the gauge-invariant and gauge-dependent variables, f, j^0, $\vec{j}$ and θ, φ, $\vec{a}$, respectively, and their semi-decoupled system of equations in the manner described in the previous section.

Since the angular variable $\theta(\tau, \vec{u})$ need not be single-valued, the contour integral of its spacetime gradient $\partial_\mu\theta$ could *a priori* lead to an arbitrary real winding number $L[C]$ in correspondence with the specific contour C being considered, such that

$$\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu\theta = -L[C]. \tag{46}$$

Hence for an arbitrary contour C, the integrated (covariant) London equations read

$$\Phi[C] = L[C] - \frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, j_\mu = L[C] + \frac{1}{2\pi}\oint_C \mathrm{d}u^\mu \frac{1}{f^2} J_\mu. \tag{47}$$

In the case of a contour C which only lies in space directions, this general expression reduces to the following relation for the corresponding magnetic flux:

$$\Phi[C] = L[C] + \frac{1}{2\pi}\oint_C \mathrm{d}\vec{u}\cdot\vec{j} = L[C] - \frac{1}{2\pi}\oint_C \mathrm{d}\vec{u}\cdot\frac{\vec{J}}{f^2}. \tag{48}$$

When considering a contour C at infinity in the case of an unbounded superconducting domain, this expression leads to the well known property of magnetic flux quantization at infinity, since the electromagnetic current density contribution then vanishes while $L[C]$ must be an integer for the topological reason recalled in the Introduction.

In fact, for all choices of contours, including those at a finite distance, the values for $L[C]$ are restricted by the necessary single-valuedness of the order parameter ψ, which constrains the lack of single-valuedness in the phase parameter θ in correlation with the fact that the function f is real but with a sign which could vary throughout spacetime, while its overall sign is left unspecified. As explained in the Introduction, to best consider that issue let us now imagine taking the scalar field ψ around the closed contour C. Given the associated winding number $L[C]$ defined in (46), under transport around C the phase variable θ is shifted by the quantity $(-2\pi L[C])$:

$$\theta \xrightarrow{C} \theta - 2\pi L[C] \qquad \mathrm{e}^{\mathrm{i}\theta} \xrightarrow{C} \mathrm{e}^{-2\mathrm{i}\pi L[C]}\,\mathrm{e}^{\mathrm{i}\theta}. \tag{49}$$

In turn, since the order parameter ψ (which is assumed not to be vanishing throughout spacetime) must be single-valued under such a transformation, the function f itself must transform according to

$$\psi \xrightarrow{C} \psi \qquad f \xrightarrow{C} \mathrm{e}^{2\mathrm{i}\pi L[C]} f. \tag{50}$$

However, since the function f is to take real values only (but not necessarily positive ones only), consistency of such transformations thus restricts the possible winding numbers $L[C]$ for any contour C to integer as well as half-integer values only. In the case of integer winding number values, the function f recovers its original sign after transport around the corresponding closed contours, while the phase θ is then shifted by an even multiple of π. In the case of half-integer values, however, f changes sign under such a transformation, in correlation with the shift by an odd multiple of π in the phase θ. Nevertheless in either case, the complex scalar field ψ remains single-valued. Note also that if one artificially restricts the function f to take positive values only, as is done in the usual classification theorems [8], only integer winding number configurations survive the analysis with the possibility of a vanishing order parameter at discrete locations only but not in a continuous fashion such as inside some domain walls.

Given this topological characterization of solutions, in the case of a half-integer winding number $L[C]$ the function f must necessarily vanish and change sign an odd number of times when taken around the closed contour C, while in the case of an integer winding number f may vanish and change sign an even number of times, including zero of course. Moreover in the case of a nonvanishing winding number $L[C]$ (be it integer or half-integer) for a contour C which shrinks to a point, necessarily the order parameter ψ and thus the function f must vanish at that point, since the phase variable θ is then ill-defined at that point being shifted by $(-2\pi L[C])$ when taken around that point. Hence by continuity, given a contour C of nonvanishing winding number $L[C]$ and a surface S with C as a boundary, the general solutions to the GLH equations may be characterized by having the order parameter ψ vanish at some point on S as well as on a series of continuous lines lying within S and emanating from that point, the number of such lines being even in the case of an integer winding number and odd for a half-integer one. Viewed in three space dimensions, such configurations correspond to vortices lying along some space-like one-dimensional curve and to which an even or odd number of two-dimensional finite or semi-infinite domain walls are attached at one of their edges according to whether the value of the vortex winding number is integer or half-integer, respectively[9].

In order to characterize the possible networks of vortices and domain walls, let us turn again to the gauge invariance properties of the model. As emphasized previously, the flux $\Phi[C]$ is both Lorentz and gauge invariant. In fact, under gauge transformations, see (42), the winding number contribution to $\Phi[C]$ transforms as

$$L'[C] = -\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu \theta' = -\frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu\, [\theta + \chi] = L[C] - \frac{1}{2\pi}\oint_C \mathrm{d}u^\mu\, \partial_\mu \chi \tag{51}$$

while the contribution from the current j_μ is gauge invariant by itself. Hence, it would appear that the winding numbers of any solution could be shifted away through some appropriate gauge transformation χ whose contour integrals exactly cancel the winding number contribution to the flux $\Phi[C]$ for all contours C. Such gauge transformations, however, leading to topology change in the gauge and Higgs fields, are not allowed. They would necessarily correspond to functions χ which are not well defined throughout spacetime, possessing singularities precisely cancelling those of the vortex solutions. Since only gauge transformations χ which are regular

[9] Note that in the general case, the winding number $L[C]$ is a function of time for any specific fixed contour C in space. This function is piece-wise constant, with integer or half-integer discontinuities at those specific instants when integer or half-integer vortices leave or enter the closed contour C.

and single-valued throughout spacetime are acceptable, the winding numbers $L[C]$ as well as the fluxes $\Phi[C]$ are indeed gauge-invariant physical observables for whatever choice of contour C, $L[C] = L'[C]$.

On the other hand, for solutions of nonvanishing winding number, the phase variable θ must necessarily possess specific singularities at some points in spacetime, since some of its contour integrals are nonvanishing even when shrinking to a point which then corresponds to the location of a vortex. Nevertheless, the regular component of θ may be changed at will through arbitrary gauge transformations (42) parametrized by regular functions χ, without changing the winding numbers of θ but affecting the gauge potential a_μ accordingly. This remark suggests that it should be possible to gauge away entirely any regular contribution to the phase variable θ, leaving over only the singular contributions responsible for all winding numbers encoded in θ while modifying the gauge potential appropriately through the relevant gauge transformation χ. To show how this can be achieved, let us assume to have constructed a specific function θ_0 whose winding numbers for all spacetime contours C reproduce exactly all those of θ given a specific vortex configuration solving the GLH equations (such a function will be constructed explicitly in the next section). Since the difference $(\theta_0 - \theta)$ is a regular function which is well defined throughout spacetime, let us then apply the gauge transformation of parameter $\chi = \theta_0 - \theta$, leading to

$$\theta' = \theta_0 \qquad a'_\mu = a_\mu - \partial_\mu\theta + \partial_\mu\theta_0. \tag{52}$$

However, given the second-order equation in (39), the transformed gauge potential a'_μ is such that

$$\partial^\mu a'_\mu = \partial_\mu a^\mu - \partial_\mu\partial^\mu\theta + \partial_\mu\partial^\mu\theta_0 = \partial_\mu j^\mu + \partial_\mu\partial^\mu\theta_0. \tag{53}$$

Since the current j^μ is gauge invariant, this last relation for a'_μ remains invariant under all further gauge transformations χ such that $\partial_\mu\partial^\mu\chi = 0$. Whether only the trivial solution $\chi = 0$ obeys this equation depends on the choice of boundary conditions for the electromagnetic sector of the system at infinity[10]. Whatever the case may be in this respect, the important conclusion is thus as follows. Given a specific configuration of bound vortices and domain walls whose topological structure is encoded in the function θ_0, one may impose the 'vortex gauge-fixing condition'[11]

$$\partial_\mu a^\mu = \partial_\mu j^\mu + \partial_\mu\partial^\mu\theta_0. \tag{54}$$

The solution for the gauge-dependent variables of the system is then given by

$$\theta = \theta_0 \qquad a_\mu = \partial_\mu\theta_0 + j_\mu. \tag{55}$$

Depending on the choice of boundary conditions at infinity, the vortex gauge fixing (54) may not be complete, in which case both θ and a_μ are defined only up to those gauge transformations whose parameter function χ also obeys the equation $\partial_\mu\partial^\mu\chi = 0$, but leading nevertheless to a construction which is consistent with the above expressions both for θ and a_μ, since both these quantities are then gauge transformed accordingly. Hence, the vortex gauge-fixing condition (54) together with the knowledge of the function θ_0 solves completely the GLH equations in the sector of gauge-dependent variables θ and a_μ. Note that when nontrivial gauge transformations such that $\partial_\mu\partial^\mu\chi = 0$ exist, the arbitrariness that such a situation leads to in terms of θ and a_μ may, in fact, be absorbed into the choice of function θ_0 which then also suffers the same physically irrelevant ambiguity.

[10] We refrain from being more specific on this point since one often considers the application of external electric and magnetic fields outside the superconductor.

[11] Note that this gauge-fixing condition is Lorentz invariant, and that for time-independent axially symmetric configurations, it reduces to the Coulomb-London gauge-fixing condition $\vec{\partial} \cdot \vec{a} = 0$ (see the next section).

2.4. The function θ_0 and the double-sheeted covering of the plane

In order to understand how to construct the function θ_0 in the general situation, let us first restrict to time-independent configurations in the planar case. The system is thus assumed to have been dimensionally reduced to two flat space dimensions, with the time-independent fields dependent only on the two coordinates of that plane, the magnetic field $\vec{b}$ purely transverse to that plane and the electric field $\vec{e}$ and charge distribution j^0 vanishing. Choosing coordinates (normalized to the length scale λ) which are either cartesian, with $x = u^1$, $y = u^2$, or polar, with $u = \sqrt{x^2 + y^2}$, $\phi = \mathrm{Arctan}(y/x)$ and the specific evaluation $-\pi \leqslant \phi \leqslant +\pi$, it is useful to introduce the complex combinations

$$z = x + \mathrm{i}y = u\mathrm{e}^{\mathrm{i}\phi} \qquad z^* = x - \mathrm{i}y = u\mathrm{e}^{-\mathrm{i}\phi}. \tag{56}$$

Let us then consider a specific collection of K vortices described as follows. Each of these vortices, labelled by $k = 1, 2, \ldots, K$, has integer or half-integer winding number $L_k = N_k/2$, N_k being an even or odd integer, respectively. The total winding number of the configuration is $L = N/2$ with $L = \sum_{k=1}^{K} L_k$ and $N = \sum_{k=1}^{K} N_k$. Furthermore, each of these vortices has a position in the plane corresponding to the complex parameter $z_k = x_k + \mathrm{i}y_k = u_k\mathrm{e}^{\mathrm{i}\phi_k}$. In order to construct the function θ_0 associated to this vortex configuration, let us also introduce for each value of $k = 1, 2, \ldots, K$ a collection of N_k complex functions $\theta_{k,n_k}(z)$ labelled by $n_k = 1, 2, \ldots, N_k$ and dependent only on the complex variable z. Then, a choice of function $\theta_0(x, y) = \theta_0(u, \phi)$ which reproduces all the contour integrals associated to this vortex configuration is given by the following expression:

$$\theta_0 = \frac{1}{4}\mathrm{i}\sum_{k=1}^{K}\sum_{n_k=1}^{N_k} \ln\left[\frac{\mathrm{e}^{-\mathrm{i}\theta_{k,n_k}(z)}\,(z - z_k)}{\mathrm{e}^{+\mathrm{i}\theta^*_{k,n_k}(z^*)}\,\left(z^* - z_k^*\right)}\right] \tag{57}$$

where a specific evaluation of the complex logarithmic function is to be chosen (for example, with its branch cut along the real negative axis in the plane, namely at $\phi = \pm\pi$ for $\ln(z)$).

The introduction of the index n_k taking a total of N values is related to the interpretation discussed in the Introduction which views all possible configurations as built from 1/2-vortices. Each value for the index n_k refers to each such 1/2-vortex as a basic building block. However, one may also write, up to some specific integer multiple of π to be added on the rhs,

$$\theta_0 = \frac{1}{2}\mathrm{i}\sum_{k=1}^{K} L_k \ln\left[\frac{z - z_k}{z^* - z_k^*}\right] + \frac{1}{4}\sum_{k=1}^{K}\sum_{n_k=1}^{N_k}\left[\theta_{k,n_k}(z) + \theta^*_{k,n_k}(z^*)\right]. \tag{58}$$

Since the double sum on k and n_k in the rhs of this expression defines a real function $\chi(x, y)$ which trivially satisfies the equation $\partial_\mu\partial^\mu\chi = 0$, this term may be gauged away altogether, leaving over only the first simple sum over k. Hence, when taking that degree of freedom into account which allows to gauge away all functions $\theta_{k,n_k}(z)$, there is no loss of generality in the following specific choice, for instance, for the function θ_0 associated to any vortex configuration:

$$\theta_0 = \frac{1}{2}\mathrm{i}\sum_{k=1}^{K} L_k \ln\left[\frac{z - z_k}{z^* - z_k^*}\right] \qquad \mathrm{e}^{\mathrm{i}\theta_0} = \prod_{k=1}^{K}\left[\frac{z - z_k}{z^* - z_k^*}\right]^{-L_k/2}. \tag{59}$$

As a particular example, consider a single vortex of winding number L at the centre of the plane. One then has

$$\mathrm{e}^{\mathrm{i}\theta_0} = \mathrm{e}^{-\mathrm{i}L\phi} \tag{60}$$

so that the order parameter takes the general form

$$\psi(u, \phi) = f(u, \phi)\,\mathrm{e}^{-\mathrm{i}L\phi}. \tag{61}$$

Having chosen to work with the angular range $-\pi \leqslant \phi \leqslant +\pi$, the phase factor $\theta = \theta_0 = -L\phi$ of this configuration has a branch cut along the negative x axis at $\phi = \pm\pi$ and starting at the origin, while $f(u, \phi)$ must then change sign on that branch cut for a half-integer winding number L. A similar discussion applies in general, by considering the neighbourhood of any vortex in the plane. Hence, whenever half-integer vortices are present (which are then necessarily bound onto the edges of domains walls), these branch cut properties show that the function f, in fact, defines a double covering of the plane, or a finite domain of it in the case of a bounded superconductor, with branch points at the positions of these vortices in such a way that $f(u, \phi)$ be continuous on the specific double-sheeted covering of the plane which is associated to the considered configuration of bound vortices and domain walls. Clearly, the same general picture also applies to integer vortices bound onto domain walls. It is only in the case of isolated integer vortices that the function $f(u, \phi)$ is, in fact, continuously single-valued on the plane (x, y) itself, rather than on some double covering of it, in which case it is indeed justified to assume that it be always positive (or negative) everywhere, with the exception of isolated points where it vanishes. Nonetheless, given the arbitrary overall sign for the function $f(u, \phi)$, it remains more appropriate, even in such a case, to view that function as defining a double-sheeted covering of the (x, y) plane in which the two sheets cross one another only at single points corresponding to the positions of the isolated integer vortices.

By allowing all the degrees of freedom hidden in the polar parametrization of the order parameter, $\psi = f\mathrm{e}^{\mathrm{i}\theta}$, to manifest themselves in a way made consistent by $U(1)$ gauge invariance, we see that any solution to the GLH equations is associated to a specific double covering of the plane or a finite domain of it. In particular, the unspecified overall sign of the function f corresponds to a choice of sheet in this double-sheeted covering. When only isolated integer vortices are involved, these two sheets become almost disconnected by touching only at isolated points, thereby allowing the sign of f to be fixed to remain either positive (or negative) as has always been assumed implicitly until now. But when domain walls are involved as well, then necessarily the two sheets become entangled in a topologically nontrivial manner precisely along these domain walls. Finally, in the case of a planar domain of finite extent, some of the lines of vanishing f values may extend all the way up to the boundary of the domain, while this general picture is compounded even further by the possibility of annular current flows surrounding the vortices, and thus adding further concentric closed structures in an almost periodic fashion to the double covering of the planar domain, with the function f changing sign at the boundaries of successive annular flows.

As a last remark concerning the function θ_0 as constructed above for time-independent planar configurations, note that it satisfies the equation $\partial_\mu \partial^\mu \theta_0 = 0$. Hence in such a situation the vortex gauge-fixing condition (54) reduces to $\partial_\mu a^\mu = \partial_\mu j^\mu$, namely $\partial_x a^1 + \partial_y a^2 = \partial_x j^1 + \partial_y j^2$. For an axially symmetric time-independent configuration, this is the Coulomb-London gauge-fixing condition $\vec{\partial} \cdot \vec{a} = 0$.

Let us now return to the general situation of a collection of integer and half-integer vortices bound onto the edges of a collection of domain walls, all in a time-dependent fashion and moving in a three-dimensional space. It is then possible to define[12] a curvilinear system of coordinates associated to a specific foliation of spacetime, such that two of the coordinates define space-like surfaces which are locally transverse to each of the vortices throughout space, with the third space-like coordinate locally transverse to these surfaces, and finally the fourth time-like coordinate tranverse (in terms of the Minkowski metric) to all three space-like coordinates. At each instant in time, it is thus possible to view the vortex system as being obtained from the previous planar description by appropriately bending these planes and by

[12] Such a coordinate system is not unique.

stacking them on top of one another in a transverse direction in such a manner that all vortices and domain walls remain continuous throughout space. The coordinates in the planes locally transverse to all vortices play the same role as the cartesian or polar coordinates considered in the planar case above. Hence, it is possible to construct again the function θ_0 in this general case using exactly the same functional expression as above, but this time in terms of the appropriate curvilinear coordinates. The only difference is then that the complex coordinates $z_k(\tau)$ and functions $\theta_{k,n_k}(z,\tau)$ become functions of time as well (with the functions $\theta_{k,n_k}(z,\tau)$ amenable to being gauged away as before). This is how the topological structure and time dependency of a given vortex and domain wall configuration may be encoded into the function θ_0, from which the solutions for $\theta = \theta_0$ and $a_\mu = \partial_\mu\theta_0 + j_\mu$ may be obtained in the vortex gauge $\partial_\mu a^\mu = \partial_\mu j^\mu + \partial_\mu\partial^\mu\theta_0$. Note that in this general case, the function θ_0 no longer obeys the equation $\partial_\mu\partial^\mu\theta_0 = 0$.

This concludes the general discussion of the topological classification of all solutions to the coupled Maxwell and GLH equations. The sector of gauge-dependent variables, namely θ and a_μ, is determined in terms of a function θ_0 (possibly defined up to some gauge transformation) which encodes the gauge-invariant topological structure characterized through all the winding numbers associated to a given configuration of vortices and domain walls. The sector of gauge-invariant variables, namely f (up to an overall sign), j^0 and $\vec{j}$, is determined by solving the coupled GLH and Maxwell equations (32) and (33), subjected to specific boundary conditions which include the 'global boundary conditions' (47) obtained from the electromagnetic flux values $\Phi[C]$ for all possible closed contours C in spacetime, corresponding to the global integrated London equations and dependent on the vortex configuration through its winding numbers. This last set of equations and boundary conditions is totally gauge invariant, which is an advantage when solving this system. Furthermore, one should keep in mind that the function f defines a double-sheeted covering of the plane with branch points at vortices. Finally, the (free) energy of such configurations may be obtained from the general expression (40).

It is interesting to consider a few sample situations corresponding to a single straight vortex of winding number L placed at the centre of the infinite plane, for which the winding phase factor is given by $\mathrm{e}^{-\mathrm{i}L\phi}$. In the case of an isolated integer vortex, the physical configuration is invariant under rotations around the vortex, since only the order parameter $\psi(u,\phi) = f(u)\mathrm{e}^{-\mathrm{i}L\phi}$ is not cylindrically symmetric because of its winding phase factor while no domain wall is attached to the vortex. Consequently, at a fixed distance u from the vortex, the order parameter retains a fixed norm $|f(u)| < 1$, so that when taking the order parameter around the contour of radius u from $\phi = -\pi$ to $\phi = +\pi$, the values reached by ψ in the complex plane all lie on the circle of radius $|f(u)| < 1$ which is wound L times starting from $\psi = \mathrm{e}^{\mathrm{i}\pi L} f(u) = (-1)^L f(u)$ and ending back at the same point on that circle. This circle also lies somewhere in between the absolute minimum $|\psi| = 1$ and the local maximum $\psi = 0$ of the surface in the shape of the bottom of a wine bottle which is defined by the Higgs potential $\left(1 - |\psi|^2\right)^2$.

Consider now an integer vortex, say with $L = 1$, but this time bound to an even number of domain walls. Rotational invariance being broken, when going around the contour of radius u the order parameter $\psi(u,\phi) = f(u,\phi)\mathrm{e}^{-\mathrm{i}L\phi}$ no longer retains a constant norm. Rather, starting from one of the domains walls where $\psi = 0$, $f = 0$, the order parameter moves in a continuous fashion from the top of the Higgs potential at $\psi = 0$ down to some maximal value for the norm $|\psi| < 1$ and back to the local maximum at $\psi = 0$ by following a closed path which passes through the latter point as often as there are domain walls bound to the vortex. When there are two domain walls, a single such loop is followed from top to bottom and back twice in the same direction. When there are four domain walls, for instance, the entire closed circuit lying on the surface of the Higgs potential is in the shape of a four-leaved clover whose centre is at the top of the Higgs potential, and so on for still larger even numbers of domain walls.

Similarly in the case $L = 1/2$ with a single domain wall, the order parameter $\psi(u, \phi) = f(u, \phi)\mathrm{e}^{-\mathrm{i}L\phi}$ follows only once a closed loop running from the top to some lowest point on the Higgs potential surface and back. When there are three domain walls bound to the $L = 1/2$ vortex, a three-leaved clover shape is obtained, and so on. Taking a $L = 3/2$ vortex to which three domain walls are bound, again a single loop running from top to bottom and back is followed three times, while when only one domain wall is bound to the $L = 3/2$ vortex the closed circuit is followed only once and reaches the top of the Higgs potential only once, but it then also explores surroundings lying on opposite sides of the top of the Higgs potential by following a spiralling path which leaves and returns at the top in a direction perpendicular to that of its lowest point.

Incidentally, these simple examples indicate that it is unlikely that a general topological classification of solutions, based on homotopy group considerations as it applies to isolated integer vortices being based on the manifold of vacuum configurations of the Higgs potential, could be developed for the wider classes of solutions described in this paper.

3. Two dimensions and BPS bounds

3.1. Two-dimensional reduction

For the remainder of the paper, the discussion is restricted to time-independent configurations in the plane. Since when considering bounded domains in the plane these will have the topology of either a disk or an annulus, the appropriate choice of coordinates is the polar one with the u and $-\pi \leqslant \phi \leqslant +\pi$ variables. A disk topology will have a radius b, namely $u_b = b/\lambda$ given our choice of normalized units, while an annular topology will have inner and outer (normalized) radii $u_a = a/\lambda$ and $u_b > u_a$, respectively.

An arbitrary time-independent configuration in the plane is represented by the following set of quantities. The electric field $\vec{e}$, scalar gauge potential φ and charge distribution j^0 all vanish. The magnetic field $\vec{b}$ is perpendicular to the plane with a single component $b(u, \phi)$, such that $\vec{b}(u, \phi) = (0, 0, b(u, \phi))$. Finally, the vectors $\vec{j}(u, \phi)$ and $\vec{a}(u, \phi)$ only have radial and azimuthal components, namely $\vec{j}(u, \phi) = \big(j_u(u, \phi), j_\phi(u, \phi), 0\big)$ and $\vec{a}(u, \phi) = \big(a_u(u, \phi), a_\phi(u, \phi), 0\big)$. All these nonvanishing quantities are functions of u and ϕ only, but of neither the coordinate transverse to the plane nor of time, as are then also the order parameter $\psi(u, \phi)$ and its polar decomposition variables $f(u, \phi)$ and $\theta(u, \phi)$. Note that this restriction on the decomposition of the vector potential $\vec{a}$ is consistent with the vortex gauge-fixing condition (54). It also proves useful to introduce the following notation:

$$j(u, \phi) \equiv j_u(u, \phi) \qquad g(u, \phi) \equiv u j_\phi(u, \phi). \tag{62}$$

Given these specific symmetry restrictions (namely translation invariance in time and in the direction transverse to the plane, or in other words parallel to the straight vortices and domain walls), it is immediate to determine the form of the system of equations. For the GLH equation (32), one finds

$$\left[\, u\partial_u u\partial_u + \partial_\phi^2 \,\right] f = \left[(uj)^2 + g^2\right] f - \kappa^2 u^2 \left(1 - f^2\right) f \tag{63}$$

while the inhomogeneous Maxwell equations (33) reduce to (in first-order form)

$$u\partial_u b = f^2 g \qquad \partial_\phi b = -f^2 u j \tag{64}$$

where the magnetic field (transverse component) is given by the local second London equation

$$b = \frac{1}{u}\partial_u g - \frac{1}{u}\partial_\phi j. \tag{65}$$

The current conservation equation, which follows from (64) of course, reads

$$u\partial_u\left(f^2uj\right)+\partial_\phi\left(f^2g\right)=0. \tag{66}$$

The set of coupled differential equations (63) and (64) is subjected to a series of boundary conditions. First, the 'global boundary conditions', corresponding to the global London equations, are

$$\Phi[C]=L[C]+\frac{1}{2\pi}\oint_C \mathrm{d}\vec{u}\cdot\vec{j} \tag{67}$$

where C is any possible choice of closed contour C in the plane, $L[C]$ is the winding number of the quantum phase variable $\theta(u,\phi)=\theta_0(u,\phi)$ (in the vortex gauge) associated to that contour, and $\Phi[C]$ is the magnetic flux (normalized to the quantum of flux Φ_0) through the surface S in the plane bounded by C

$$\Phi[C]=\frac{1}{2\pi}\int_S \mathrm{d}u\,\mathrm{d}\phi\, ub(u,\phi). \tag{68}$$

In particular, for a circular contour centred on the origin $u=0$ of the plane and of radius u_0, we have the condition

$$\frac{1}{2\pi}\int_0^{u_0}\mathrm{d}u\int_{-\pi}^{+\pi}\mathrm{d}\phi\, ub(u,\phi)=\Phi[u_0]=L[u_0]+\frac{1}{2\pi}\int_{-\pi}^{+\pi}\mathrm{d}\phi\, g(u_0,\phi) \tag{69}$$

where $L[u_0]$ stands, of course, for the associated winding number in $\theta(u_0,\phi)$.

In the case of the infinite plane, the local boundary conditions at infinity which complete those in (67) and which are required for finite energy configurations are such that $f(u,\phi)$ approaches either one of the two values $f=\pm1$, while the quantities $j(u,\phi)$ and $g(u,\phi)$ must both vanish, since they directly define the components of the electromagnetic current density $\vec{J}=-f^2\vec{j}$ in the plane.

In the case of a bounded domain in the plane, there may exist an applied external magnetic field $\vec{b}_{\rm ext}$ solely transverse to the plane with a component $b_{\rm ext}$. Based on Maxwell's equations in vacuum outside the bounded domain, in the specific instance of such a geometry for $\vec{b}(u,\phi)$ it follows that the magnetic field $\vec{b}(u,\phi)$ must retain the constant homogeneous value $\vec{b}_{\rm ext}$ throughout space in that region. Consequently, given the remaining boundary conditions relevant to $f^2\vec{j}$ and $\vec{\partial}f$, it follows that in the case of the disk topology the required local boundary conditions which complete those in (67) are

$$b(u_b,\phi)=b_{\rm ext}\qquad j(u_b,\phi)=0\qquad (\partial_u f)(u_b,\phi)=0. \tag{70}$$

In the case of the annular topology, the same boundary conditions apply, of course, at the outer boundary $u=u_b$, while at the inner boundary $u=u_a$, one must also have

$$b(u_a,\phi)=b_a\qquad j(u_a,\phi)=0\qquad (\partial_u f)(u_a,\phi)=0 \tag{71}$$

where b_a represents the value of the constant magnetic field (transverse component) within the hole of the annulus. The value for b_a is to be determined from the total magnetic flux $\Phi[u_a]$ through that hole which is to be expressed in terms of the angular integral of $g(u_a,\phi)$ and the winding number $L[u_a]$ in the quantum phase $\theta(u_a,\phi)$ on the inner boundary, as given in (69) with $u_0=u_a$. The fact that $b(u,\phi)$ must be ϕ-independent on the boundaries is consistent with the determination of $b(u,\phi)$ inside the superconductor in terms of $j(u,\phi)$ and $g(u,\phi)$ and with the boundary conditions on $j(u,\phi)$ (see (64)).

Due to the invariance of the considered configurations under translations transverse to the plane, the three-dimensional integral in (40) defining the free energy is infinite. One should rather consider the expression for the free energy per unit of length in the transverse direction, while the overall normalization factor in (40) may also be absorbed in that choice, leading in those units to the expression

$$\mathcal{E} = \int_0^\infty \mathrm{d}u \int_{-\pi}^{\pi} \mathrm{d}\phi\, u \left\{ [b - b_{\text{ext}}]^2 + (\partial_u f)^2 + \frac{1}{u^2}\left(\partial_\phi f\right)^2 + f^2\left(j^2 + \frac{1}{u^2}g^2\right) + \frac{1}{2}\kappa^2\left(1 - f^2\right)^2 \right\}. \tag{72}$$

Note that the constant term $(-\kappa^2/2)$ has not been added to the integrand of this expression, keeping in mind the possible infinite planar topology in some applications (for the finite disk and annular topologies, this term will be included later on). Even though the radial integration extends throughout the infinite plane, it is only within the domain of the superconductor that f, j and g do not vanish, while in the outer region of the superconductor (if u_b is finite) we always have $b = b_{\text{ext}}$, so that there is no contribution to the free energy for $u > u_b$ in the cases of the disk and annular topologies (there is a contribution for $u < u_a$ in the annulus case). Actually, when evaluated for a specific solution to the differential equations (63) and (64), the expression for the free energy as given in (72) also reduces to

$$\mathcal{E} = \int_0^\infty \mathrm{d}u \int_{-\pi}^{\pi} \mathrm{d}\phi\, u \left\{ [b - b_{\text{ext}}]^2 - \frac{1}{2}\kappa^2 f^4 + \frac{1}{2}\kappa^2 \right\}. \tag{73}$$

The above equations and boundary conditions define the problem to be solved. The only information missing is that which provides the topological structure of the vortex and domain wall configuration. This structure is characterized by the winding numbers $L[C]$ which appear in the global boundary conditions (67). The gauge-dependent details of that topological structure are specified through the function $\theta_0(u,\phi)$, which in the vortex gauge (54) may be expressed as in (59), and in terms of which the quantities $\theta(u,\phi) = \theta_0(u,\phi)$ and $\vec{a}(u,\phi) = \vec{j} - \vec{\partial}\theta_0$ are then obtained. Nevertheless, it is only the gauge-invariant content of θ_0, through the winding numbers $L[C]$, which is involved in the resolution of the remaining quantities $f(u,\phi)$, $j(u,\phi)$ and $g(u,\phi)$, which are to be determined from the above differential equations and boundary conditions, while also keeping in mind the double-sheeted covering of the plane which is associated to the vortex and domain wall configuration being considered and which is defined by the function $f(u,\phi)$.

As a final point, consider the case of the infinite plane or the disk with a vortex of winding number $L_0 \neq 0$ placed exactly at its centre $u = 0$, and a circular contour of radius u_0 surrouding that vortex. In the limit that the contour shrinks to the origin, since the magnetic flux $\Phi[u_0]$ must vanish (the magnetic field being finite at $u = 0$), the azimuthal component of the current density $\vec{J} = -f^2\vec{j}$ at that point must be such that

$$\lim_{u_0\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, g(u_0,\phi) = -L_0 \tag{74}$$

which necessarily requires

$$g(0,\phi) = -L_0. \tag{75}$$

On the one hand, since $j_\phi(u,\phi) = g(u,\phi)/u$, this result shows that the current $\vec{j}$ is necessarily ill-defined at the location of vortices. But on the other hand, since the order parameter ψ and the function f vanish at the location of the vortex, while the azimuthal component of the electromagnetic current $\vec{J}$ must vanish at the same point (this is a consequence of the polar coordinate parametrization which is being used), this result also shows that the singularity in $\vec{j} = -\vec{J}/f^2$ is just mild enough to be screened by the zero in ψ or f in such a way that the physical current $\vec{J}$ remains well-defined and finite throughout space as it should, even at the location of vortices.

3.2. BPS bounds

The resolution of the GLH equations, even in the plane, requires a numerical analysis. Nevertheless, it is possible to establish some general properties for solutions in particular situations. Namely, for specific values of the scalar self-coupling κ or λ_0, BPS lower bounds [9, 10] apply to the energy of vortex configurations.

To this aim, let us consider the free energy expressed in (72) in the absence of any external magnetic field, $b_{\rm ext} = 0$, and for the disk and infinite planar topologies[13]. Through integration by parts and the identification of the geometric meaning of the induced surface terms, one finds that

$$\begin{aligned}\mathcal{E} = \eta\left[2\pi\sum_{k=1}^{K} L_k + \oint_{\partial\Omega} d\vec{u}\cdot\left(1-f^2\right)\vec{j}\right] + \int_\Omega d^2\vec{u}\Bigg\{\Bigg[\left(\frac{1}{u}\partial_u g - \frac{1}{u}\partial_\phi j\right)\\ -\frac{1}{2}\eta\left(1-f^2\right)\Bigg]^2 + \left[\partial_u f + \eta\frac{1}{u}fg\right]^2 + \left[\frac{1}{u}\partial_\phi f - \eta f j\right]^2\Bigg\}\\ +\frac{1}{2}\left(\kappa^2 - \frac{1}{2}\right)\int_\Omega d^2\vec{u}\left(1-f^2\right)^2 \qquad (76)\end{aligned}$$

where $\eta = \pm 1$ is some choice of sign to be specified presently, Ω stands for the planar domain in which $f \neq 0$, namely that of the superconductor, and $\partial\Omega$ for the boundary of that domain. Finally, $L = \sum_{k=1}^{K} L_k$ is the total winding number of the vortex configuration lying within the domain Ω.

The remarkable feature of this relation is that, with the exception of the surface term at $\partial\Omega$ and the very last term which depends on the self-coupling κ, this expression gives the energy of any solution in terms of a sum of integrated positive quantities in such a way that the topological content of the configuration is made explicit through the total winding number L. When attempting to minimize the value of the energy by setting to zero the positive integrated quantities, thus leading to the following first-order differential equations:

$$b = \frac{1}{u}\partial_u g - \frac{1}{u}\partial_\phi j = \frac{1}{2}\eta\left(1-f^2\right) \qquad \partial_u f = -\eta\frac{1}{u}fg \qquad \partial_\phi f = \eta u f j \qquad (77)$$

it is immediate to verify that these equations then also imply the second-order equations in (63) and (64), but only if the coupling takes the specific value $\kappa = \kappa_{\rm c}$ with $\kappa_{\rm c} = 1/\sqrt{2}$. In other words, when the scalar self-coupling takes the critical value $\kappa = \kappa_{\rm c}$, the system of first-order equations (77) integrates the second-order GLH equations of motion (the issue of boundary conditions will be addressed shortly). Note that it is for the same critical coupling $\kappa_{\rm c}$ that the very last term in (76) does not contribute to the energy of such configurations.

In order to fix the choice of sign for $\eta = \pm 1$, recall that the lowest possible value taken by $\mathcal{E}$ as defined in (72) is zero, corresponding to the trivial solution $b(u,\phi) = b_{\rm ext}$ (here $b_{\rm ext} = 0$), $f(u,\phi) = \pm 1$, $j(u,\phi) = 0$ and $g(u,\phi) = 0$ in the absence of any vortex in the plane. Consequenly, all other solutions must necessarily possess a strictly positive energy value, which dictates the following choice of sign

$$\eta = \mathrm{sign}\left(\sum_{k=1}^{K} L_k\right) = \mathrm{sign}\, L. \qquad (78)$$

Given this choice for η and the specific value $\kappa = \kappa_{\rm c}$, the energy of any solution to the first-order equations (77) (which then also obey the second-order ones) is given by

$$\mathcal{E} = 2\pi\left|\sum_{k=1}^{K} L_k\right| + (\mathrm{sign}\, L)\oint_{\partial\Omega} d\vec{u}\cdot\left(1-f^2\right)\vec{j}. \qquad (79)$$

[13] The annulus case is left to the reader.

However, it is only in the case of the infinite plane that the surface term on the boundary $\partial\Omega$ does not contribute, since the boundary conditions are then such that both $(1-f^2)$ and $\vec{j}$ vanish at infinity. Furthermore, it is clear that these boundary conditions are also consistent with the first-order equations (77). Hence finally, one concludes that all solutions to the first-order equations (77), which also obey the second-order GLH equations and the appropriate boundary conditions in the infinite plane when $\kappa = \kappa_c$, saturate the BPS value which determines their energy or mass in terms of a topological invariant, namely the total winding number of that configuration,

$$\mathcal{E} = 2\pi \left| \sum_{k=1}^{K} L_k \right| = 2\pi |L|. \tag{80}$$

Conversely, it has been shown [8] that under both the assumptions of an everywhere positive function f and possessing only a discrete set of zeroes of integer positive degree, all the solutions to the second-order GLH equations and boundary conditions in the infinite plane with $\kappa = \kappa_c$ also solve the first-order equations (77) with $\eta = \operatorname{sign} L$, thus saturating the BPS value and showing [15] that such vortices do not possess an interaction energy which would depend on their relative positions.

However, when relaxing these restrictions on the function f, and thereby allowing the possibility of half-integer vortices and bound domain walls in the infinite plane, one must conclude that such configurations cannot solve the first-order equations (77) and thus cannot saturate the BPS value (80). Indeed, the first of these first-order equations would imply that on the surface of vanishing order parameter, $f = 0$, within any domain wall, the magnetic field b would retain a constant value of $\eta/2$ irrespective of the distance between the vortices which are bound onto the edges of that domain wall and of their winding numbers. Such a property seems very unlikely, not only because of this lack of dependency of the magnetic field on these parameters, but also because the magnetic energy density contribution to $\mathcal{E}$ would then grow linearly with that distance, a fact which would be inconsistent with the saturation of the BPS value. On the other hand, we also know that the further a domain wall is stretched (provided the vortices bound onto its two edges no longer overlap), the more the condensation energy stored in the domain wall contributes to $\mathcal{E}$ in an almost linear fashion as well, again a property which would be incompatible with the saturation of the BPS value (80). Hence, it must be concluded that even for the critical value $\kappa = \kappa_c$, only isolated integer vortex solutions to the GLH equations in the infinite plane both solve the first-order equations (77) and saturate the BPS value, while any other vortex configuration including domain walls cannot share these properties. Only the usual isolated vortex configurations of integer winding number obey the self-dual first-order equations (77) at the critical coupling $\kappa = \kappa_c$. Nonetheless, configurations with domain walls bound to vortices in the infinite plane do obey the following BPS strict lower bound:

$$\mathcal{E} > 2\pi \left| \sum_{k=1}^{K} L_k \right| = 2\pi |L|. \tag{81}$$

Indeed considering (76) again, the surface term at infinity vanishes in the infinite plane while the first two-dimensional volume integral is strictly positive since such configurations cannot obey the first-order equations (77) (the last term vanishes for $\kappa = \kappa_c$).

Let us now turn to the disk of finite radius u_b. One is then led to the conclusion that the BPS lower bound $\mathcal{E} \geqslant 2\pi|L|$ does not apply for solutions to the second-order GLH equations with $\kappa = \kappa_c$, even for isolated integer vortices. The first-order equations (77) are, in fact, incompatible with the required boundary conditions (70) [18]. Even when $b_{\text{ext}} = 0$, these equations and boundary conditions together would imply that

$$f(u_b, \phi) = \pm 1 \qquad g(u_b, \phi) = 0 \qquad (\partial_\phi f)(u_b, \phi) = 0. \tag{82}$$

If $\sum_{k=1}^{K} L_k = L \neq 0$, such conditions can be met only if the boundary of the disk is at infinity, $u_b \to \infty$, thereby recovering the previous discussion in the infinite plane. This conclusion on its own, however, would still not exclude the relevance of the BPS lower bound to solutions in the disk. In fact, even had the first-order equations been consistent with the boundary conditions, there cannot exist a BPS lower bound on the energy of any solution in a finite disk, including the case of isolated integer vortices. Indeed, the surface contribution $\oint_{\partial\Omega} \mathrm{d}\vec{u} \cdot (1 - f^2)\vec{j}$ does not vanish in such a situation, and could *a priori* carry whatever sign depending on the details of the distribution of vortices within the disk.

The situation with respect to possible BPS lower bounds having been understood in the case of the critical coupling $\kappa = \kappa_c$, let us now consider the general expression (76) relevant whatever the value for κ. For couplings larger than the critical one, $\kappa > \kappa_c$, and configurations in the infinite plane thus leading to a vanishing surface term at infinity, the energy always obeys the BPS strict lower bound in (81) whatever the solution to the GLH equations. Indeed, on the one hand, the very last contribution in (76) proportional to $(\kappa^2 - 1/2)$ is then always strictly positive while, on the other hand, the remaining integrated positive terms cannot vanish either since otherwise this would require the first-order equations (77) (with $\eta = \mathrm{sign}\, L$) to be obeyed by solutions to the GLH equations, which is excluded when $\kappa \neq \kappa_c$. In the case of the finite disk with $\kappa > \kappa_c$, again no BPS lower bound may be given because of the nonvanishing contribution of either sign from the surface term on the disk boundary. Values for the energy $\mathcal{E}$ for configurations in the disk with $\kappa > \kappa_c$ could *a priori* be smaller as well as larger than the BPS value $2\pi \left| \sum_{k=1}^{K} L_k \right| = 2\pi |L|$ depending on the distribution of vortices.

Finally, when $\kappa < \kappa_c$, the same conclusion as to the absence of a BPS bound must be drawn, whether for vortex configurations in the infinite plane or the disk, since in that case, the contribution of the very last term in (76) is always strictly negative.

4. Half-integer vortex solutions

Having argued that the basic entities from which to build general configurations are 1/2-vortices, 1/2-domain walls and annular current flows, this section considers the situation of a single vortex at the centre of either a disk or an annulus of finite radius. Even though no exact analytical solutions exist, we shall try to gain some insight into the nature of such solutions by using approximations, and then turn to the results of a modest first attempt at a numerical analysis of half-integer vortices.

For solutions with half-integer winding number, here we only consider the case of a single domain wall bound on the vortex, and further assume that this domain wall lies along a radius of the disk or annulus. Even though such configurations break axial symmetry, this breaking is then kept to a minimum, which remains covariant under rotations around the centre of the disk or annulus mapping such solutions into one another without changing their energy. This restriction is also in keeping with the fact that the natural tension which domain walls possess is such that their lowest energy configuration must be straight not only in the direction transverse to the plane, but also within that plane.

4.1. Annular vortices

In order to argue for the possibility of annular vortices of integer as well as half-integer winding number, we use the approach of [18] which uncovered the existence of such configurations in the isolated integer case by considering the emergence of these solutions from the trivial solution $\psi = 0$ at the normal–superconducting phase transition. Namely, we restrict to the disk topology with a single vortex of winding number L at its centre, in the absence of any external magnetic field. The solution which describes the system at the phase transition is then given by

$$g(u,\phi) = -L \qquad j(u,\phi) = 0 \qquad b(u,\phi) = 0 \qquad f(u,\phi) = 0 \tag{83}$$

so that the order parameter vanishes identically, $\psi(u,\phi) = 0$, with a vanishing energy for the state, $\mathcal{E} = 0$ (for the remainder of the paper, the subtraction constant $(-\kappa^2/2)$ is being included in the definition of the free energy (72)), and a nonvanishing winding number nonetheless.

This solution may also be obtained by considering a rescaling by a real parameter f_0 of the order parameter, such that

$$\psi(u,\phi) = f_0\,\tilde{\psi}(u,\phi) \qquad f(u,\phi) = f_0\,\tilde{f}(u,\phi) \tag{84}$$

and then taking the limit $f_0 \to 0$ in the GLH equations and relevant boundary conditions. Doing so, one finds that the function $\tilde{f}(u,\phi)$ satisfies the linearized GLH equation

$$\left[u\partial_u u\partial_u + \partial_\phi^2\right]\tilde{f} = \left[(uj)^2 + g^2\right]\tilde{f} - \kappa^2 u^2 \tilde{f}. \tag{85}$$

Given the solution in (83), as well as the requirement that $\tilde{f}(u,\phi)$ should vanish at the origin when $L \neq 0$ since this is the position of the vortex where $f(u,\phi)$ must vanish, the linearized GLH equation possesses a single solution. The case of an isolated vortex of integer winding number has been discussed in [18]. Let us restrict our attention here to that of a half-integer value for the winding number L of a single vortex which is bound onto a single domain wall lying along a radius of the disk (the generalization to more domain walls and integer winding numbers is immediate, as the reader will realize).

Under these specific assumptions, and the fact that the equation for $\tilde{f}(u,\phi)$ in the limit $f_0 \to 0$ is linear, it is natural to consider the following separation of variables:

$$\tilde{f}(u,\phi) = \tilde{f}(u)\sin(\phi/2) \tag{86}$$

which clearly displays a cut at $\phi = \pm\pi$, having assumed the domain wall to be at $\phi = 0$ (the branch cut for $\tilde{f}(u,\phi)$ thus also lies at $\phi = \pm\pi$). The solution for $\tilde{f}(u)$ is then the Bessel function of the first kind of order $\alpha = \sqrt{L^2 + 1/4}$:

$$\tilde{f}(u) = \left(\frac{2}{\kappa}\right)^{\alpha} \Gamma(1+\alpha) J_\alpha(\kappa u) \tag{87}$$

where the normalization is chosen such that the lowest-order term in a series expansion in u is $\tilde{f}(u) \simeq u^\alpha$. As is well known, such a Bessel function has an almost periodic oscillatory behaviour with an amplitude which asymptotically decreases as $1/\sqrt{u}$ at large radii.

Consequently, as soon as the parameter f_0 is slightly turned on away from zero, this oscillatory pattern hidden in the solution $f(u,\phi) = 0$ will emerge from the vanishing condensate, and lead to a pattern of concentric annular current flows of approximately constant width π/κ (in units of λ) surrounding the half-integer vortex of winding number L which is bound onto the edge of a single domain wall (only the index α of the Bessel function as well as the periodicity of the trigonometric function in ϕ change when more domain walls are involved). Of course, the boundary conditions at $u = u_b$ are then no longer satisfied, so that both the parameter f_0 as well as the remaining functions j and g must be adjusted in order to obtain a solution of negative energy to the GLH equations, thereby also pushing outwards the pattern of concentric annular current flows without essentially changing their width [18].

Nevertheless, the existence of the Bessel function solution to the linearized GLH equation at the phase transition provides a very strong argument for the existence in finite domains of annular vortex solutions of arbitrary half-integer winding number bound onto the edges of an arbitrary number of domain walls. The existence of such solutions without domain walls has indeed been established along such lines for integer vortices in [18].

Even though it is impossible to obtain an exact analytic solution when turning on the parameter f_0, even in the form of a power series expansion, it is interesting to consider the first-order corrections that this implies for the quantities j, g and b, without including the corrections implied for $f(u,\phi)$ itself. To lowest order in u, and still assuming only a single domain wall, one finds, when also imposing the current conservation equation,

$$\begin{aligned}
uj(u,\phi) &= \frac{L}{2\alpha}\frac{\cos(\phi/2)}{\sin(\phi/2)} \\
g(u,\phi) &= -L + \frac{1}{2}b_0u^2 - \frac{L}{4\alpha(\alpha+1)}f_0^2u^{2(\alpha+1)}\sin^2(\phi/2) - \frac{L}{2\sin^2(\phi/2)}\ln(u/u_0) \\
b(u,\phi) &= b_0 - \frac{L}{2\alpha}f_0^2u^{2\alpha}\sin^2(\phi/2) \\
f(u,\phi) &= f_0u^\alpha\sin(\phi/2)
\end{aligned} \tag{88}$$

where b_0 stands for the value of the magnetic field at the vortex $u = 0$ and u_0 for an arbitrary integration constant. These expressions are at least indicative of the singularities in $j(u,\phi)$ and $g(u,\phi)$ not only at the position of the vortex (a fact which was already demonstrated previously) but also on the surface of vanishing order parameter at $\phi = 0$, but in such a way that the physical electromagnetic current $\vec{J} = -f^2\vec{j}$ remains well-defined and regular everywhere nonetheless, including the quantum tunnel effect through the domain wall close to the vortex.

4.2. The thin annulus limit

Another instance for which it is possible to gain insight into half-integer winding number solutions in a particular limit is that when the width of the annular topology vanishes, the outer radius being kept fixed, namely the limit $u_a \to u_b$. Let us assume that a single winding number L is trapped in the centre of the annulus of width $\Delta u = u_b - u_a$ and outer radius u_b. Consequently, the value b_a of the magnetic field inside the hole must be such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, g(u_a,\phi) = \frac{1}{2}u_a^2b_a - L. \tag{89}$$

In the limit that the annulus becomes infinitely thin, $u_a \to u_b$, the magnetic field both within the superconductor as well as inside the hole takes the value b_{ext} of the applied field. Moreover in that limit we shall also take the approximation $j(u,\phi) = 0$, the annulus having no radial extent anymore, even though such a restriction may prove to be inconsistent with the current conservation condition. In this limit, not only does the above global boundary condition imply a specific restriction on the function $g(u_b,\phi)$ in terms of L and b_{ext}, but also the free energy (72) then reduces to the expression

$$\mathcal{E} \simeq \frac{\Delta u}{u_b}\int_{-\pi}^{\pi}\mathrm{d}\phi\left\{(\partial_\phi f)^2 + f^2g^2 + \frac{1}{2}\kappa^2u_b^2f^4 - \kappa^2u_b^2f^2\right\}. \tag{90}$$

In order to address the problem of minimizing the free energy and thus solving the GLH equations in the thin annulus limit, let first consider the case of an integer winding number L in the absence of any domain wall. Because of the axial symmetry of such a configuration, no ϕ-dependency arises for the functions b, g and f, and one is left only with the following relations (all these quantities being evaluated at $u = u_b$, of course):

$$g = \frac{1}{2}u_b^2b_{\text{ext}} - L \qquad \mathcal{E} \simeq 2\pi\frac{\Delta u}{u_b}\left[f^2g^2 + \frac{1}{2}\kappa^2u_b^2f^4 - \kappa^2u_b^2f^2\right]. \tag{91}$$

It is immediate to show that the free energy is minimized for the following values:

$$f^2 = 1 - \frac{1}{\kappa^2 u_b^2} g \qquad \mathcal{E} \simeq -\frac{\pi\,\Delta u}{u_b}(\kappa^2 u_b^2)\left[1 - \frac{1}{\kappa^2 u_b^2}\left(L - \frac{1}{2}u_b^2 b_{\text{ext}}\right)^2\right]^2. \tag{92}$$

In turns out that this quartic dependency of $\mathcal{E}$ on the applied field b_{ext} provides a rather good approximation to exact numerical solutions for annuli of finite width Δu small compared to u_b, since it presents a turn-over point precisely at the values of b_{ext} where this expression vanishes. This property is very close to the behaviour of the exact solutions, while it is not reproduced by the quadratic dependency which is usually discussed in textbooks [16, 17]. Whatever the relevance of this latter remark, the important point is that when considered for all possible integer values L, the graphs for the free energy $\mathcal{E}$ as a function of b_{ext} all cross one another in succession before their turn-over points at $\mathcal{E} = 0$ provided that $\kappa u_b > 1/2$ and for values such that

$$\frac{b}{\xi} = \kappa u_b > \frac{1}{2}: \qquad \frac{1}{2}u_b^2 b_{\text{ext}} = k + \frac{1}{2} \qquad \mathcal{E}(k) \simeq -\frac{\pi\,\Delta u}{u_b}(\kappa^2 u_b^2)\left[1 - \frac{1}{4\kappa^2 u_b^2}\right] \tag{93}$$

k being an arbitrary positive, negative or zero integer. This, of course, is the experimentally observed behaviour as well, namely the Little–Parks effect [16, 17], the crossing points occurring for a magnetic flux through the annulus equal to a half-integer multiple of the quantum of flux Φ_0 (note also the lower bound on the annulus radius, $b > \xi/2$). Furthermore, the absolute minimum of the free energy is reached for

$$\frac{1}{2}u_b^2 b_{\text{ext}} = k \qquad \mathcal{E}_{\text{min}} \simeq -\frac{\pi\,\Delta u}{u_b}(\kappa^2 u_b^2) \tag{94}$$

while the absolute maxima are reached for

$$\frac{1}{2}u_b^2 b_{\text{ext}} = k \pm \kappa u_b \qquad \mathcal{E} \simeq 0. \tag{95}$$

Let us now consider the case of a half-integer winding number L, which should thus also include some domain wall structure inside the thin annulus. The axial symmetry of the equations being broken, all quantities $b(\phi)$, $g(\phi)$ and $f(\phi)$ become ϕ-dependent. The minimization of the free energy leads in this case to the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} f = f g^2 - \kappa^2 u_b^2 (1 - f^2) f \tag{96}$$

while the function $g(\phi)$ is constrained by the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, g(\phi) = \frac{1}{2}u_b^2 b_{\text{ext}} - L. \tag{97}$$

On the other hand, the current conservation equation reduces in the present limit to

$$\partial_\phi (f^2 g) = 0 \tag{98}$$

so that

$$g(\phi) = \frac{C}{f^2(\phi)} \qquad \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} f = \frac{C^2}{f^3} - \kappa^2 u_b^2 (1 - f^2) f \tag{99}$$

with C some integration constant.

First in the case that this constant vanishes, $C = 0$, we necessarily have $g(\phi) = 0$, which implies that such a situation is obtained only for values of the applied field such that

$$\frac{1}{2}u_b^2 b_{\text{ext}} = k + \frac{1}{2} = L \tag{100}$$

k being an arbitrary integer. These are precisely the values for which the energy of the thin annulus is degenerate for two consecutive integer winding numbers (provided $\kappa u_b > 1/2$). The solution for the order parameter is then

$$f(\phi) = \pm \tanh\left(\frac{\kappa u_b}{\sqrt{2}}\phi\right) \tag{101}$$

leading to a value for the free energy of this configuration given by

$$\mathcal{E} \simeq -\frac{\pi \Delta u}{u_b}(\kappa^2 u_b^2)\left\{1 - \frac{2\sqrt{2}}{\pi\kappa u_b}\tanh\left(\frac{\pi\kappa u_b}{\sqrt{2}}\right)\left[1 - \frac{1}{3}\tanh^2\left(\frac{\pi\kappa u_b}{\sqrt{2}}\right)\right]\right\}. \tag{102}$$

This value always lies above that $\mathcal{E}(k)$ for the integer winding number case at the same magnetic field values, whatever the value for $\kappa u_b > 1/2$.

The solution found for $f(\phi)$ displays a cut at $\phi = \pm\pi$, in accordance with the half-integer winding number, and vanishes at $\phi = 0$, indicating the presence of a single domain wall in the thin annulus. However, the condensate $|\psi(\phi)|^2 = f^2(\phi)$ presents a discontinuity in its angular derivative at the cut at $\phi = \pm\pi$, since we have

$$f^2(\phi) = \tanh^2\left(\frac{\kappa u_b}{\sqrt{2}}\phi\right) \qquad \frac{\mathrm{d}}{\mathrm{d}\phi}f^2(\phi) = \sqrt{2}\kappa u_b \tanh\left(\frac{\kappa u_b}{\sqrt{2}}\phi\right)\left[1 - \tanh^2\left(\frac{\kappa u_b}{\sqrt{2}}\phi\right)\right]. \tag{103}$$

Such a discontinuity is not physically acceptable, and is a consequence of the thin annulus limit. Nevertheless, this approximate solution to the full GLH equations provides some interesting insight into the properties of half-integer vortex solutions as well as an explicit illustration of the double-sheeted covering of the plane associated to a general solution.

Turning now to the situation when the integration constant C does not vanish, the differential equation (99) to be solved, is also that of a nonrelativistic particle of unit mass moving on the real line $-\infty < f < +\infty$ and subjected to the potential

$$V(f) = \frac{C^2}{2f^2} + \frac{1}{2}\kappa^2 u_b^2\left(f^2 - \frac{1}{2}f^4\right) \tag{104}$$

whose conserved energy K_f is thus

$$\left(\frac{\mathrm{d}}{\mathrm{d}\phi}f\right)^2 + V(f) = K_f. \tag{105}$$

Hence, there cannot exist a solution for $f(\phi)$ passing through the value $f = 0$ for any finite value of the integration constant K_f, since the potential $V(f)$ diverges at that point. An infinite value for K_f would rather be required, but this in turn is physically unacceptable since in such a case the angular variation $\mathrm{d}f/\mathrm{d}\phi$ of the order parameter would be infinite at all points except possibly at those where $f = 0$, thus also leading to an infinite energy. Hence we must conclude that no solution to the above equations may be found when $C \neq 0$.

The physical interpretation of these results is as follows. Having taken the thin annulus limit $u_a \to u_b$ with the restriction that $j(\phi) = 0$, this is consistent with the current conservation equation only for integer winding numbers without domain walls, since the axial symmetry of such solutions even for an annulus of finite width is such that $j(u, \phi) = 0$ in any case. However for half-integer winding numbers or integer vortices bound to domain walls, this limit is such that all states are squeezed out from the annulus by acquiring an infinite energy. It is only when the applied field b_{ext} takes the particular values at which two consecutive integer winding states without domain walls become degenerate (which requires $\kappa u_b > 1/2$), that the states of half-integer winding number and with a single domain wall across the annulus retain a finite energy, but yet develop a discontinuity in the angular variation of the order parameter.

4.3. Half-integer vortices in the disk

Let us finally turn to some numerical solutions to the GLH equations (63) and (64) in the disk. The numerical resolution of these differential equations presents some genuine challenges. Even in the case of axially symmetric configurations, namely a single isolated giant vortex of integer winding number at the centre of the disk, the direct numerical integration of the then ordinary differential equations for $f(u)$, $g(u)$ and $b(u)$ ($j(u) = 0$ in that case) are quite delicate. These equations read

$$u\frac{\mathrm{d}}{\mathrm{d}u}b(u) = f^2(u)g(u) \qquad u\frac{\mathrm{d}}{\mathrm{d}u}\left[u\frac{\mathrm{d}}{\mathrm{d}u}\right]f(u) = g^2(u)f(u) - \kappa^2u^2\left(1 - f^2(u)\right)f(u) \tag{106}$$

with of course $ub(u) = \mathrm{d}g(u)/\mathrm{d}u$ and the boundary conditions

$$\begin{aligned} g(0) = -L: &\qquad \left(\frac{\mathrm{d}}{\mathrm{d}u}f\right)(0) = 0 \quad \text{if} \quad L = 0 \qquad f(0) = 0 \quad \text{if} \quad L \neq 0 \\ b(u_b) = b_{\mathrm{ext}}: &\qquad \left(\frac{\mathrm{d}}{\mathrm{d}u}f\right)(u_b) = 0 \end{aligned} \tag{107}$$

L being the integer winding number. The main reason for difficulties in this case is, on the one hand, that the equations are nonlinear (the cubic term in f^3 in the GLH equation makes the stability of any numerical integration problematic), and on the other hand, that the above boundary conditions are, in fact, delocalized on the two boundaries of the disk, namely at $u = 0$ and at $u = u_b$. Hence, whether integrating inwards or outwards, half of the initial values must be adjusted in order to meet the required boundary conditions at the other boundary. Nevertheless, it is still possible to manage such difficulties, and even obtain the annular vortex solutions although the precision requirements on the initial values are then critical [18].

In the case of configurations which are no longer axially symmetric, namely those involving either isolated integer vortices which are no longer at the centre of the disk, or domain walls, or both, these difficulties are much greater, since the nonlinear equations are then partial differential ones in two variables for a larger number of quantities to be integrated, and with not only delocalized local boundary conditions but also the global boundary conditions corresponding to the integrated London equations for all contours in the plane. Furthermore, as we have seen, the current $\vec{j}(u, \phi)$ is then also singular at the positions of vortices and inside the domain walls. These singularities at the location of a vortex which is at the centre of the disk can be handled as above in the axially symmetric case, since we then have $j(0, \phi) = 0$ and $g(0, \phi) = -L_0$. Unfortunately, such a simplification does not apply to vortices away from the origin $u = 0$ nor to domain walls.

The alternative to integrating numerically the partial differential equations (63) and (64) subjected to the local and global boundary conditions, is to numerically minimize the free energy (72) using a steepest descent method of one type or another. Since the free energy coincides, up to a sign, with the action in the case of time-independent configurations, the minimization of the energy is equivalent to solving the equations of motion. Most of the difficulties mentioned above may then be circumvented by an appropriate choice of lattice discretization. Lattice sites are regularly spaced both in u and in ϕ, in the ranges $0 \leqslant u \leqslant u_b$ and $-\pi \leqslant \phi \leqslant \pi$, the odd numbers of steps being distinct for each variable. When present, the domain wall lies along the direction $\phi = 0$. The quantities f, j and g are defined at each lattice site, and are varied each in turn and at each lattice site in succession according to the following steepest descent procedure.

Consider in general a system with degrees of freedom q_i ($i = 1, 2, \cdots, N$) for which a quantity $S(q_i)$ needs to be minimized, assuming it possesses an absolute minimum. An efficient procedure is defined by the iterative algorithm

$$q_i' = q_i - S_{ij}^{-1}(q_i)S_j(q_i) \qquad S_i(q_i) = \frac{\partial S}{\partial q_i}(q_i) \qquad S_{ij}(q_i) = \frac{\partial^2 S}{\partial q_i \partial q_j}(q_i). \tag{108}$$

In particular, in the case of a positive definite quadratic form $S(q_i)$, this procedures converges to the absolute minimum in only one iteration, whatever the initial configuration q_i. In our case, however, in order to avoid having to invert the large matrix S_{ij}, this algorithm is applied at each lattice site in succession for each of the variables g, j and f separately, until the norm of the gradient S_i of the free energy in configuration space convergences sufficiently close to zero. Nonetheless, the whole minimization procedure requires significant computational power. This is, in a few words, the general method used to obtain the results of this section and in section 5.2.

Simulations of only modest computer time have been run so far, lacking great precision. The only results described in this section apply to the following choice of parameters:

$$u_b = 3 \qquad \kappa = 1. \tag{109}$$

The lattice discretization uses the polar coordinate parametrization with 10 intervals in the radial variable $0 \leqslant u \leqslant u_b$ and 10 intervals in the angular range $0 \leqslant \phi \leqslant \pi$. The only configurations considered involve either a single integer vortex with $L = 0, 1, 2$ without a domain wall at the centre of the disk, or else a single half-integer vortex with $L = 1/2, 3/2$ also at the centre of the disk and then bound onto only one domain wall. This domain wall is taken to be straight and to extend all the way to the boundary of the disk along the radius at $\phi = 0$. Consequently, all these configurations possess specific symmetry properties under $\phi \to -\phi$, which is an advantage in order to reduce by half the calculation labour.

The coarse-grained discretization used may imply a rather poor precision. Since the same procedure is applied to all configurations, one may hope that whatever the numerical uncertainties, they would all contribute in the same direction so that at least the general properties, if not the precise numerical values, are to be trusted. In the case of axially symmetric configurations with integer winding number, the angular integration becomes trivial and a far more precise resolution, either by minimizing the free energy or by integrating the ordinary differential equations of motion using a fourth-order Runge–Kutta method, becomes possible. The comparison between the different approaches then shows that the steepest descent method used in the half-disk leads to values for the free energy which are precise to 1% to 2% but which all deviate from their actual values in the same direction as expected. Hence, the same quality should apply to the results for the configurations with $L = 1/2$ and $L = 3/2$.

Figure 1 presents the results for the free energy of the disk in these different vortex and domain wall configurations as a function of the applied external magnetic field b_{ext}. The well known behaviour of the energy for the $L = 0, 1, 2$ configurations is reproduced. The curves for the solutions of winding numbers $L = 1/2$ and $L = 3/2$ are totally new, and are quite noteworthy. First, at the values of b_{ext} where the curves of integer L cross each other, the energy values for the half-integer winding numbers always lie above those for the integer winding numbers, in agreement with the analysis in the thin annulus limit. Second, the magnetic field dependency of the energy for half-integer winding numbers is less pronounced than for the integer ones, and extends to larger values of b_{ext} before hitting the phase transition line at $\mathcal{E} = 0$. Nevertheless, these curves always lie above at least one of those of integer winding numbers (including $L > 2$ not represented), so that the lowest energy states, at least for the geometry and the κ value considered, always correspond to some giant vortex state without a domain wall. Third, the $L = 1/2$ and $L = 3/2$ curves lie much closer to one another than do the integer winding number ones. In particular, it is quite remarkable that at $b_{\text{ext}} = 0$, both

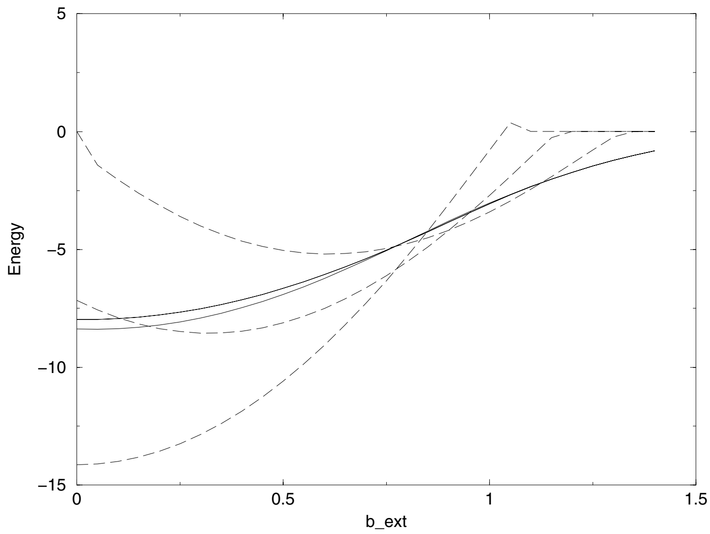

Figure 1. The free energy $\mathcal{E}$ of the disk as a function of the applied external magnetic field, for $u_b = 3$ and $\kappa = 1$. The vortex configurations are those discussed in section 4.3, with the vortices at the centre of the disk and having winding numbers $L = 0, 1, 2$ and $L = 1/2, 3/2$, including in the latter two cases a single domain wall extending up to the disk boundary. The three dashed lines correspond, from bottom to top, to the states with $L = 0, 1, 2$ in the same order, while the two continuous lines correspond, in the same order again, to the states with $L = 1/2, 3/2$. For further details, see section 4.3.

half-integer winding number curves lie below all the giant vortex ones without domain wall for $L \neq 0$.[14]

Clearly, this is only one example in a very specific case. It would be extremely interesting to see how these different features change when varying the geometry of the disk and the value for κ, when moving the vortices around within the disk, when allowing the domain walls to end on some other vortex inside the disk, when including more domain walls, and so on. All these issues should be relevant to the understanding of the magnetization properties and dynamics of mesoscopic superconducting disks and loops in varying electromagnetic fields.

Even though not displayed, the numerical solutions for the configurations considered above also confirm the fact that there is an electromagnetic current flow which quantum tunnels through the domain wall close to the vortex position for the $L = 1/2, 3/2$ winding number cases. Within the core of these vortices, the magnetic field is, of course, not axially symmetric, but it even presents a shape such that its values first increase when moving out from the centre of the vortex before decreasing towards the disk boundary. This volcano-like shaped surface also possesses a small valley-like depression in the direction of the domain wall, along which the order parameter $|f|$ takes, of course, smaller values and thus lets the magnetic field penetrate further in and decrease less rapidly as well (see (64)).

In spite of the modesty of these first numerical results, they do establish the existence of half-integer winding number solutions to the GLH equations, bound to domain walls, and demonstrate the potential these new configurations have to contribute to the static and dynamic properties of any physical system which is described, even if only in an effective way, by the Abelian $U(1)$ Higgs model.

[14] In this respect, it would be interesting to see where the curve for the $L = 3/2$ with three domain walls would lie.

5. Bound 1/2-domain walls

5.1. The solitonic limit

Bound 1/2-domain walls were argued to provide the basic entities in terms of which any configuration in the infinite plane could be viewed as a particular combination. Before considering in the next section the results of a modest attempt at a numerical resolution, here we shall present a specific bound 1/2-domain wall solution to the GLH equations (63) and (64) in the infinite plane.

Even though this solution possesses an infinite energy, it may, in fact, be viewed as corresponding to a bound 1/2-domain wall which is infinitely stretched in a specific direction so that the two 1/2-vortices bound onto its edges are at infinity. It is this infinite length of the domain wall which explains its infinite energy, of course. In spite of this fact, however, the solution is interesting in that it illustrates another example of the double-sheeted covering of the plane as well as some of the properties of bound 1/2-domain walls in a particularly simple limit, while also demonstrating that the energy of domain walls is linear in their length when the vortices at their edges no longer overlap.

To describe this specific solution, it is best to consider a system of cartesian coordinates x and y in the infinite plane (normalized to λ), and to assume that the domain wall is aligned, say, with the x axis, so that the configuration is also invariant under translations parallel to that axis. Hence, all variables must be independent of x. Furthermore, given the physical interpretation in which the 1/2-vortices are at infinity, one should expect that both the magnetic field $b(y)$ as well as the current $\vec{j}(y)$, and thus also the electromagnetic current density $\vec{J}(y) = -f^2(y)\vec{j}(y)$, vanish everywhere in the plane,

$$b(y) = 0 \qquad \vec{j}(y) = \vec{0}. \tag{110}$$

These restrictions are clearly consistent with the current conservation equation, while the GLH equation for the order parameter then reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} f(y) = -\kappa^2 \left(1 - f^2(y)\right) f(y). \tag{111}$$

This latter equation is of course well known, and possesses solitonic solutions running from one vacuum $f = -1$ of the Higgs potential $(1 - f^2)^2$ to the other $f = +1$. In the present case, the solution we are interested in is that which corresponds to a single domain wall separating two regions of the plane in which each of these two expectations values are reached at infinity.

Hence, the infinitely stretched bound 1/2-domain wall is described by the one soliton or antisoliton configuration

$$f(y) = \pm \tanh\left(\frac{\kappa}{\sqrt{2}} y\right). \tag{112}$$

Note how this solution takes values of opposite signs on both sides of the x axis, the choice of $\pm$ sign in this expression corresponding to a choice of sheet in the double-sheeted covering of the plane which is defined by these one soliton and antisoliton configurations. Similarly, a $N > 1$ soliton (or antisoliton) solution would correspond to N such infinitely stretched bound 1/2-domain walls all lying parallel to one another in the infinite plane. However, depending on the value of N, such states are not topologically stable and decay either to a single bound 1/2-domain wall if N is odd, or else to the vacuum configuration $|f(y)| = 1$ if N is even.

Being translationally invariant along a specific direction in the plane, these configurations possess an energy (72) which is infinite. Nevertheless, their energy per unit of length Δx along their symmetry axis, chosen here to be the x axis, is finite, and for the one soliton configuration[15] is given by

$$\frac{\mathcal{E}}{\Delta x} = \frac{4\sqrt{2}}{3}\kappa. \tag{113}$$

Hence in the limit of an infinite separation of the two edges of the bound 1/2-domain wall, the energy does grow linearly with that distance. The amount of condensation energy (no magnetic energy density is present) stored in the domain wall which is of constant thickness is proportional to its length.

5.2. *The bound* 1/2*-domain wall in the disk*

Let us now consider the numerical solution for a bound 1/2-domain wall in the disk, using minimization of the free energy through steepest descent. Only the specific instance of a vanishing external magnetic field will be considered, $b_{\text{ext}} = 0$. Such an analysis touches directly onto the issue of the stability of the ANO vortex of winding number $L = \pm 1$, so that we shall, in fact, consider for different values of κ the dependency of the energy (72) as a function of the separation between the two $L = \pm 1/2$ vortices bound onto the edges of such a domain wall.

It was suggested in the Introduction that there might exist a critical value $\kappa_{1/2}$ marking the boundary between the situation where, on the one hand, the bound 1/2-domain wall would be stable for a specific streched length (function of κ), and on the other hand, where it would always collapse into the $L = \pm 1$ ANO vortex. Even though different arguments could be put forward, based on the BPS considerations of section 3.2, only a detailed numerical analysis of the problem can answer this issue of the stability of the ANO vortex as a function of the coupling κ (and of the external field b_{ext}). This section presents the results of a first attempt towards a resolution, which, unfortunately, proves to be still too modest to enable any conclusion. In fact, a full-fledged dedicated computationally intensive numerical analysis is required to resolve this question, which we hope to address elsewhere.

The specific configuration which has been analysed is as follows. Each of the $L = 1/2$ vortices is placed at an equal distance d (in units of λ) and on opposite sides from the centre of the disk, while the straight domain wall joining them is aligned with the corresponding diameter. Because of the four-fold symmetry of the configuration, it suffices to solve the problem in only one quadrant of the disk, say for $0 \leqslant \phi \leqslant \pi/2$. On the discretized lattice, when $d \neq 0$, the two vortices are, in fact, positioned half-way between radial lattice sites, in order to avoid the singularities in the functions j and g at the vortex position. The lattice discretization used 20 intervals in the radial direction $0 \leqslant u \leqslant u_b$ and 16 intervals in the angular range $0 \leqslant \phi \leqslant \pi/2$. Finally, to avoid as much as possible effects due to the finite boundary (such as the Bean–Livingston barrier [24]) while still having a sufficiently fine-grained lattice discretization, the disk radius was set to

$$u_b = 5. \tag{114}$$

The values for the coupling κ which were considered are

$$\kappa : \qquad 0.5,\ \frac{1}{\sqrt{2}},\ 1.0,\ 1.5. \tag{115}$$

For each of these values, the steepest descent minimization of the energy was solved numerically, starting with $d = 0$, as a function of a regularly spaced series of values for the distance d, which corresponds to the bound 1/2-domain wall thus streched to a length $2d$.

[15] The calculation is also feasable if the plane is of finite extent in the y direction, $-y_0 < y < y_0$, but the boundary conditions $(\mathrm{d}f/\mathrm{d}y)(\pm y_0) = 0$ are then not satisfied.

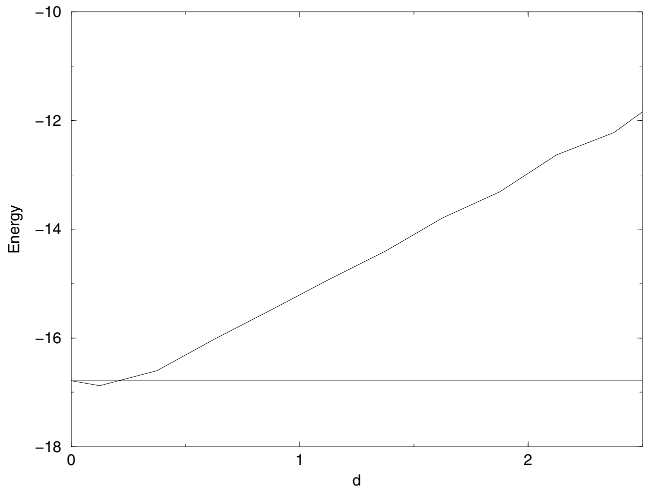

Figure 2. The energy $\mathcal{E}$ of the bound 1/2-domain wall in a disk with $u_b = 5$ and $\kappa = 1/\sqrt{2}$ as a function of the distance d to the disk centre of each of the 1/2-vortices bound onto the edges of the domain wall. For further details, see section 5.2. The horizontal line is a guide to the eye, and corresponds to the value for $d = 0$.

Figure 2 presents the results of these computations for the value $\kappa = \kappa_c = 1/\sqrt{2}$. The results for the other values of κ show exactly the same behaviour (the only difference being in the vertical scale for energy values) with, in particular, always a small dip in the energy at the first nonzero step in the values for d whatever the value for κ. The total magnetic flux $\Phi[u_b]$ was monitored and also displays a curious behaviour. When $d = 0$, one has a single $L = 1$ ANO vortex at the centre of the disk, and $\Phi[u_b]$ then takes a value very close to unity as it should, namely $\Phi[u_b] = 0.989$. However, when d increases, the values for $\Phi[u_b]$ first increase getting even closer to the value of unity, before finally decreasing as they should when the two 1/2-vortices move closer to the disk boundary. Most probably, these two results for $\mathcal{E}$ and $\Phi[u_b]$ are numerical artifacts for small values of d. Unfortunately at this stage, it is not possible to resolve that issue, which may only be addressed through a full-fledged analysis including a much finer-grained discretization which is necessarily much more computationally intensive.

Note that the numerical solution in figure 2 displays the expected linear increase in energy with separation, as soon as the bound 1/2-domain wall is sufficiently streched, namely when the two 1/2-vortices no longer overlap. The linear behaviour sets in around $d \simeq 0.5$ for all considered values of κ (recall that d is measured in units of λ).

Hence at this stage, the fascinating issue of the possible instability of the ANO vortex as a function of κ, which would then decay into a bound 1/2-domain wall streched to some length function of κ, is still completely open.

6. Conclusions and outlook

The fine-grained topological analysis which considers the transport of the complex scalar field around all possible finite contours in spacetime has thus uncovered a rich zoology of topological solutions to the Ginzburg–Landau–Higgs equations of the Abelian $U(1)$ Higgs model. Beyond the well known vortex states of integer winding number, the new configurations also include vortices of integer as well as half-integer winding number bound onto the edges of domain walls, all such vortex configurations possibly being surrounded by annular current flows in the case of bounded spatial domains. The existence and physical consistency of these new states is related to the $U(1)$ local gauge invariance of the model, leading to the characterization of these states in terms of a double-sheeted covering of the plane, in addition to its angular multicovering associated to a nonvanishing winding number. The physical properties of these solutions and of their mathematical construction have been described in the Introduction and established in the body of the paper.

Even though the numerical analysis has not yet answered some important issues, such as the identification of the lowest energy configurations for specific combinations of such states, perhaps the most intriguing of these being the possible instability of the Abrikosov–Nielsen–Olesen vortex to split into a domain wall with two vortices of winding number $L = 1/2$ bound onto its edges, the mere existence of such solutions raises a host of potential dynamical properties. It now becomes possible to imagine that under electromagnetic disturbances, the magnetic vortices described by these equations would not only move around in space but could also be stretched elastically into domain walls with vortices bound onto their edges, very much like rubber bands being pulled in different directions. Such properties may have interesting consequences, for instance, with regard to the transport of electromagnetic energy in systems obeying these equations.

Beyond their physical interest, these solutions raise other questions as well. From the mathematical physics point of view, it would be very satisfying if the stability of such topological configurations, and in particular that of the associated domain walls, could be put on a firm mathematical ground, as has been done for the usual vortex solutions [8, 12]. Even though, as pointed out previously, the usual homotopy group arguments are not of much help in this respect in the general case, this does not mean that more refined topological considerations may not be relevant to the issue, such as those which have led to the discovery of sphaleron solutions in the electroweak sector of the Standard Model of particle physics [25].

Indeed, analogous half-integer winding number sphaleron configurations have been shown to contribute to transitions between vacua of consecutive integer winding number values for the Ginzburg–Landau–Higgs model in 1 + 1 dimensions [26]. In particular, these half-integer sphalerons define saddle point configurations for the energy functional. Extrapolated to 3 + 1 dimensions, this result might be considered as establishing the instability of domain wall configurations with bound half-integer vortices, through a process in which the closed loops (described at the end of section 2.4) running to and from the top of the Higgs potential as one crosses a domain wall while circumventing a half-integer vortex, slip off that hill to fall into the valley, thereby inducing a change by $\pm 1/2$ in the winding number. Such a conclusion, however, is not warranted at this stage. Indeed, the electrodynamics in 1 + 1 dimensions is such that the electromagnetic field is not dynamical, with in particular no magnetic field which would otherwise contribute to the energetics of configurations and their dynamics. Such artifacts specific to 1 + 1 dimensions do not apply in 3 + 1 dimensions, in which, even in a static and electric field-free regime, the contributions of magnetic fields become crucial, and not only those of the scalar Higgs field through its condensation energy density as is the case in 1 + 1 dimensions. If only for this fact, one may not conclude, solely on the basis of the arguments developed in [26], that domain wall configurations define saddle points for the energy functional—a conclusion which might eventually be reached following only a dedicated and detailed analysis, which is not attempted here. As a matter of fact, the magnetic energy density contribution may also be such that domain wall configurations define actual local minima of the energy functional, since indeed the magnetic field value of vortices increases with their winding number $|L|$, again a feature which is absent in 1 + 1 dimensions.

When only the topological considerations that explain the stability properties of integer vortices within finite domains are considered, half-integer ones and their domain walls appear to share those same properties, since the winding number only takes discrete values which may change only through some singular process occuring at the boundary of the sample and thus acting from the outside. For instance, it has been shown in [27] how integer vortices move in and out of finite domains depending on the value of the applied external magnetic field. Similarly, if a 1/2-domain wall in a disk (or an annulus) is to collapse into a vortex with either $L = 0$ or $L = 1$, this would require either the $L = 1/2$ vortex bound onto its edge to be expelled from the disk along with the domain wall, or else one more $1/2$-vortex to be pulled in at its outer edge through the disk boundary. Presumably, in the same way as for integer vortices, such a process is dependent on the external magnetic field as the source or sink for the varying trapped magnetic flux, the external field possibly providing the pull required to stretch the domain wall and ensure its stability. It thus appears that the stability issue of half-integer vortices and their domain walls should be much dependent on the boundary conditions for samples of finite extent, and should thus display peculiar properties in the limit of samples of infinite extent. These issues certainly deserve a detailed analysis in order to reach a definite conclusion.

One may also consider applying (an adapted form of) the fine-grained topological analysis to other field theories known to possess topological states, such as grand unified theories with spontaneous symmetry breaking, and possibly uncover further structure in these states by , for example, lifting specific symmetry properties such as rotational invariance (among the states described in this paper, only the usual vortices possess that symmetry). Extensions to higher spacetime dimensions or differential forms of higher degree could also be envisaged.

Nonetheless, the field for which the results of this paper offer perhaps the most immediate interest is that of superconductivity. In fact, we were led into the analysis of the present issues because of our current project to develop a quantum detector for polarized particles based on superconducting loops [28]. When such a loop is polarized in an external magnetic field close enough to the switching between two successive winding number states so that some local disturbance in the magnetic field would cause the loop to switch from one state to the other and back, this nonlinear response of the loop by a significant fraction of a full quantum of flux may be picked up by an inductive circuit coupled to a SQUID. The same principle is at work in other devices using superconducting loops, albeit of larger dimensions, aimed towards the development of superconducting [29] (or) quantum [30] computers. However, in order to optimize the geometry of the loop with the particle detector application in mind, it becomes necessary to have a detailed understanding of the dynamics in a relativistic regime of the switching mechanism, which seems not to be known. Only energy-based considerations in the static regime are available for superconducting mesoscopic disks and annuli, mostly motivated by the results of [4]. The only exception may be the analysis of [27] which provides a detailed picture for how an integer vortex may move in and out a superconducting disk or loop as a function of the applied field. Now that $1/2$-vortices bound onto the edge of a $1/2$-domain wall have been uncovered from the same Ginzburg–Landau equations, the whole issue has to be reconsidered again, to see whether there are specific loop geometries and values of the Ginzburg–Landau parameter κ for which the sensitivity of the dynamical response to disturbances could be optimized, for instance, by having the $L = 1/2$ state at a lower energy than the $L = 1$ one when degenerate with the $L = 0$ state, so that magnetic flux would move in and out of such devices through a string of smaller bits than when only integer vortices are involved.

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